

# THE COMPLEXITY OF INDEX SETS OF CLASSES OF COMPUTABLY ENUMERABLE EQUIVALENCE RELATIONS

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ABSTRACT. Let  $\leq_c$  be computable reducibility on ceers. We show that for every computably enumerable equivalence relation (or ceer)  $R$  with infinitely many equivalence classes, the index sets  $\{i : R_i \leq_c R\}$  (with  $R$  non-universal),  $\{i : R_i \geq_c R\}$ , and  $\{i : R_i \equiv_c R\}$  are  $\Sigma_3^0$  complete, whereas in case  $R$  has only finitely many equivalence classes, we have that  $\{i : R_i \leq_c R\}$  is  $\Pi_2^0$  complete, and  $\{i : R_i \geq_c R\}$  (with  $R$  having at least two distinct equivalence classes) is  $\Sigma_2^0$  complete. Next, solving an open problem from [1], we prove that the index set of the effectively inseparable ceers is  $\Pi_4^0$  complete. Finally, we prove that the 1-reducibility pre-ordering on c.e. sets is a  $\Sigma_3^0$  complete pre-ordering relation, a fact that is used to show that the pre-ordering relation  $\leq_c$  on ceers is a  $\Sigma_3^0$  complete pre-ordering relation.

## 1. INTRODUCTION

Given equivalence relations  $R$  and  $S$  on the set  $\omega$  of natural numbers, we say that  $R$  is *reducible* to  $S$  (in symbols:  $R \leq_c S$ ), if there exists a computable function  $f$  such that

$$(\forall x, y)[x R y \Leftrightarrow f(x) S f(y)].$$

Given a class  $\mathcal{A}$  of equivalence relations on  $\omega$ , one says that  $R$  is  $\mathcal{A}$  *complete*, if  $R \in \mathcal{A}$ , and  $S \leq_c R$ , for every  $S \in \mathcal{A}$ . This reducibility, and this notion of completeness, have turned out to be very useful tools for measuring the complexity of equivalence relations naturally arising in mathematics, and, in particular, in computable model theory and in computability theory (where equivalence relations on structures can be viewed as relations on numbers via identification of structures with numbers, thanks to suitable indexings). For instance, Fokina, S. Friedman, Harizanov, Knight, McCoy, and Montalbán [8] show  $\Sigma_1^1$  completeness of the isomorphism relations for various familiar classes of computable structures, including computable groups, computable torsion abelian groups, computable torsion-free abelian groups, abelian  $p$ -groups. On the other hand, Fokina, S. Friedman, and Nies [7] show that other familiar equivalence relations arising from computability are  $\Sigma_3^0$  complete, including computable isomorphism of c.e. sets. (In Corollary 4.11 we give another proof of this result.) Other interesting mathematical applications of reducibility  $\leq_c$  appear in [4], [10], [11].

The reducibility  $\leq_c$ , as well as the notion of  $\mathcal{A}$ -completeness, can obviously be extended to pre-ordering relations on  $\omega$ . Ianovski, R. Miller, Ng, and Nies [12] characterize the arithmetical complexity of several pre-orders of interest to computability theory, for instance showing that almost inclusion  $\subseteq^*$ , and  $\leq_T$  on c.e. sets, are  $\Sigma_3^0$  complete.

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There is already a non-trivial literature concerning the restriction of  $\leq_c$  to computably enumerable equivalence relations (abbreviated as ceers): pioneering papers in this regard include (in chronological order) Ershov [5], Bernardi and Sorbi [2], Montagna [14], Lachlan [13], Gao and Gerdes [9], Andrews, Lempp, Miller, Ng, San Mauro, and Sorbi [1]. These papers study  $\Sigma_1^0$  complete (also called *universal*) ceers, and the degree structure of ceers under  $\leq_c$ . We investigate and classify the arithmetical complexity of some index sets of ceers. Throughout the paper, we refer to some fixed universal computable numbering  $\{R_i : i \in \omega\}$  of all ceers (see [1]), where “computable” means that the set  $\{\langle i, x, y \rangle : x R_i y\}$  is c.e., and “universal” means that for every such computable numbering  $\{S_i : i \in \omega\}$  of all ceers, there exists a computable function  $f$  such that  $S_i = R_{f(i)}$ , for all  $i$ . Extending results in [9], we give complete characterizations in the arithmetical hierarchy of the complexity of the index sets  $\{i : R_i \leq_c R\}$  (with  $R$  non-universal),  $\{i : R_i \geq_c R\}$ , and  $\{i : R_i \equiv_c R\}$ : if  $R$  has infinitely many equivalence class then all these sets are  $\Sigma_3^0$  complete, whereas if  $R$  has only finitely many equivalence classes, we have that  $\{i : R_i \leq_c R\}$  is  $\Pi_2^0$  complete, and  $\{i : R_i \geq_c R\}$  (with  $R$  having at least two distinct equivalence classes) is  $\Sigma_2^0$  complete. Solving a problem in [1], we prove that the index set of the effectively inseparable ceers is  $\Pi_4^0$  complete.

In the last section of the paper we consider  $\leq_c$  on pre-ordering relations on  $\omega$ . The literature regarding the restriction of  $\leq_c$  (as a reducibility on pre-orders) to computably enumerable pre-orders, includes, among others, the papers Pour El and Kripke [17], Montagna and Sorbi [15], and Ianovski, R. Miller, Ng, and Nies [12]: these papers are mainly dedicated to the investigation of  $\Sigma_1^0$  complete pre-orders naturally arising in logic. We prove that the pre-ordering relation  $\leq_c$  on ceers (viewed as a pre-order on their indices) is a  $\Sigma_3^0$  complete pre-ordering relation. The proof goes by first showing that the 1-reducibility pre-ordering on c.e. sets is a  $\Sigma_3^0$  complete pre-ordering relation.

**1.1. Background.** The reader is referred to [19] for all computability theoretic notions that are used, but not explicitly introduced, in this paper. For more information on ceers, their structure under  $\leq_c$ , bibliography, and even history, our basic reference is [1]. Given a ceer  $E$ , we say that a sequence  $\{E_s : s \in \omega\}$  of equivalence relations on  $\omega$  is a *computable approximation to  $E$* , if the following conditions hold: the set  $\{\langle x, y, s \rangle : x E_s y\}$  is computable;  $E_0$  is the identity equivalence relation; for all  $s$ ,  $E_s \subseteq E_{s+1}$ ; the equivalence classes of  $E_s$  are finite; there exists at most one pair  $[x]_{E_s}, [y]_{E_s}$  of equivalence classes, such that  $[x]_{E_s} \cap [y]_{E_s} = \emptyset$ , but  $[x]_{E_{s+1}} = [y]_{E_{s+1}}$  (we say in this case that the equivalence relation  $E$  *collapses  $x$  and  $y$*  at stage  $s + 1$ ); and finally  $E = \bigcup_t E_t$ . Every ceer has computable approximations; in fact we can show (see [1]) that there exists a uniform sequence  $\{R_{i,s} : i, s \in \omega\}$  of equivalence relations such that the set  $\{\langle i, x, y, s \rangle : x R_{i,s} y\}$  is computable, and for every  $i$ , the sequence  $\{R_{i,s} : s \in \omega\}$  is a computable approximation to  $R_i$ .

## 2. COMPUTING THE COMPLEXITY OF INDEX SETS OF CEERS ABOVE, BELOW, OR EQUIVALENT TO A GIVEN CEER

Index sets of classes of ceers of natural computability theoretic interest have been investigated for the first time by Gao and Gerdes [9]. Index sets of the form  $\{i : R_i \leq_c R\}$ ,  $\{i : R \leq_c R_i\}$ ,  $\{i : R_i \equiv_c R\}$ , for particular choices of  $R$ , are classified in [9] to be  $\Sigma_3^0$  complete: for instance, this is the case when  $R$  is the identity relation on the natural numbers. In this section we completely classify all index sets of this type, thus showing for instance (see Corollary 2.6) that if  $R$  is a ceer with infinitely many equivalence classes then  $\{i : R \leq_c R_i\}$ ,  $\{i : R_i \equiv_c R\}$  are always  $\Sigma_3^0$  complete, and if  $R$  has infinitely many classes and is not universal then  $\{i : R_i \leq_c R\}$  is always  $\Sigma_3^0$  complete.

**Theorem 2.1.** *Let  $R$  be a non-universal ceer with infinitely many classes. Then  $(\Sigma_3^0, \Pi_3^0) \leq_1 (\{i \mid R_i \equiv_c R\}, \{i \mid R_i \not\equiv_c R \& R_i \not\equiv_c R\})$  (where  $(\Sigma_3^0, \Pi_3^0) \leq_1 (A, B)$  means that for every  $\Sigma_3^0$  set  $C$ , there is a computable function which reduces  $C$  to  $A$ , and the complement of  $C$  to  $B$ : see [19, p. 66] for this notation).*

*Proof.* Fix a  $\Sigma_3^0$  complete set  $S := \{i \mid (\exists l)[W_{g(i,l)} = \omega]\}$ , where  $g$  is a computable function (the fact that every  $\Sigma_3^0$  set can be expressed in this way is an easy consequence of the proof of [19, Corollary IV.3.7]). We construct a function which, on input  $i$ , outputs an index of a ceer  $E$  so that if  $i \in S$  then  $E \equiv_c R$ , and if  $i \notin S$  then  $E$  and  $R$  are  $\leq_c$ -incomparable.

Given  $i$ , we describe the enumeration of the ceer  $E$  based on the enumeration of the sets  $W_{g(i,l)}$  for various  $l$ . It will be clear from the construction that an index for  $E$  can be uniformly found in  $i$ .

**Requirements and their strategies.** Given  $i$ , we have three kinds of requirements:

$$Q_l : W_{g(i,l)} = \omega \Rightarrow E \equiv_c R.$$

$$N_j : (\forall l < j)[W_{g(i,l)} \neq \omega] \Rightarrow \varphi_j \text{ does not give a reduction witnessing } E \leq_c R.$$

$$P_k : (\forall l < k)[W_{g(i,l)} \neq \omega] \Rightarrow \varphi_k \text{ does not give a reduction witnessing } R \leq_c E.$$

Let us fix some computable priority ordering on the requirements. We first describe the action taken by each requirement individually.

*Q-requirements.* A  $Q_l$ -requirement acts as follows: When initialized,  $Q_l$  is given a finite set of distinct  $E$ -equivalence classes  $[b_1]_E, \dots, [b_n]_E$  of elements created due to higher priority requirements.  $Q_l$  is also given a finite set of elements  $c_1, \dots, c_n$ .  $Q_l$  works under the assumption that the classes  $[b_i]_E$  are pairwise distinct, and the  $[c_i]_R$  are pairwise distinct. If either of these assumptions becomes incorrect,  $Q_l$  will be re-initialized.  $Q_l$  collapses all remaining elements (those created for lower priority requirements) into one class  $[d]_E$ , and, beginning with that one class, copies  $R$ , using a computable coding function  $x \mapsto a(x)$ . At every stage wherein  $\min(\omega \setminus W_{g(i,l)})$  increases,  $Q_l$  again  $E$ -collapses every new element which is created due to a lower priority requirement, to  $d$ , and continues building its copy of  $R$ ,  $E$ -collapsing codes of elements exactly as the corresponding elements are collapsed by  $R$ . If no higher priority requirement acts ever again and in fact  $\{c_1, \dots, c_n\}$  are non-equivalent in  $R$ , and  $Q_l$  acts infinitely often (as  $W_{g(i,l)} = \omega$ ), then we will argue that  $E \equiv_c R$ . Whenever  $Q_l$  acts, it restrains all elements created so far.

*N-requirements.* An  $N_j$  requirement acts as follows: We fix a universal ceer  $T$ . When initialized,  $N_j$  selects new elements  $a(0), a(1) \in \omega$ , and  $E$ -collapses these elements if and only if 0 and 1 collapse in  $T$ . If at some stage,  $\varphi_j(a(0))$  and  $\varphi_j(a(1))$  converge, and

$$0 T 1 \Leftrightarrow \varphi_j(a(0)) R \varphi_j(a(1)),$$

then  $N_j$  selects a new element  $a(2)$ ,  $E$ -collapsing (for  $m, n \leq 2$ )  $a(m)$  to  $a(n)$  if and only if  $m T n$ . If at a later stage,  $\varphi_j(a(2))$  converges and

$$(\forall n, m \leq 2)[n T m \Leftrightarrow \varphi_j(a(n)) R \varphi_j(a(m))],$$

then  $N_j$  selects a new element  $a(3)$ . The construction proceeds as such. We will argue that if no higher priority requirement re-initializes  $N_j$ , then  $N_j$  can choose only finitely many elements  $\{a(i) : i \leq k\}$ , otherwise, we would have  $T \leq_c R$  via the map  $i \mapsto \varphi_j(a(i))$ , which contradicts non-universality of  $R$ . Thus,  $\varphi_j$  can not be a reduction of  $E$  to  $R$ . Whenever  $N_j$  chooses an element  $a(k)$ , by initialization it restrains all elements  $\leq a(k)$ .

*P-requirements.* A  $P_k$ -requirement acts as follows:  $P_k$  searches for elements  $x < y \in \omega$  so that  $\varphi_k$  converges on all inputs  $\leq y$ ,  $\varphi_k(x)$  and  $\varphi_k(y)$  are not restrained by higher priority requirements (so that it is allowed to  $E$ -collapse  $\varphi_k(x)$  and  $\varphi_k(y)$ ), and  $x \not R y$ . If such are found, then  $P_k$  collapses  $\varphi_k(x)$  and  $\varphi_k(y)$  in  $E$ . If, at a later stage,  $x R y$ , then  $P_k$  is injured and begins again. If, in fact, there are only finitely many elements restrained by higher priority requirements, then some pair of elements  $x, y$  will eventually be found so that  $x \not R y$ , and either already  $\varphi_k(x) E \varphi_k(y)$ , or  $\varphi_k(x)$  and  $\varphi_k(y)$  are not restrained by higher priority requirements (since  $R$  has infinitely many classes). But then we cause  $\varphi_k(x) E \varphi_k(y)$ . This contradicts  $\varphi_k$  being a reduction of  $R$  to  $E$  after all. As  $P_k$  never minds things collapsing, it places no restraints.

*Environments for the requirements.* A  $Q_l$ -requirement uses a parameter  $\gamma_l(s) = \langle c_1, \dots, c_n \rangle$ , and values of a finite function,  $a_l^Q(x, s)$ , which approximates the function  $x \mapsto a(x)$  described in the above informal discussion for  $Q$ -requirements. An  $N_j$ -requirement uses a parameter  $a_j^N(x, s)$ , which approximates the numbers  $a(x)$  described in the above informal discussion for  $N$ -requirements. In the following, we will often omit the superscripts  $Q$ , or  $N$ , when the exact choice will be clear from the context. A  $P_k$ -requirement uses parameters  $x_k(s), y_k(s)$ , which approximate the numbers  $x, y$ , described in the above informal discussion for  $P$ -requirements. If  $\mathcal{R}$  is either a  $Q$ -requirement or an  $N$ -requirement, the construction also uses a parameter  $\rho^{\mathcal{R}}(s)$  to record the elements that  $\mathcal{R}$  wants to restrain.

**Construction.** To tackle  $N$ -requirements, we fix a universal ceer  $T$ , with computable approximations  $\{T_s\}_{s \in \omega}$ . At stage  $s$ , to *initialize* a requirement  $\mathcal{R}$  means one of the following:

- if  $\mathcal{R} = Q_l$ , then we set  $\gamma_l(s), a_l(x, s)$  to be undefined for all  $x$ ; and we set  $\rho^{Q_l}(s) = \emptyset$ ;
- if  $\mathcal{R} = N_j$ , then we set  $a_j(x, s)$  to be undefined, all  $x$ ; and we set  $\rho^{N_j}(s) = \emptyset$ ;
- if  $\mathcal{R} = P_k$ , then we set  $x_k(s), y_k(s)$  to be undefined.

At stage  $s > 0$  we say that a requirement  $\mathcal{R}$  *requires attention* if either  $\mathcal{R}$  is initialized, or

- $\mathcal{R} = Q_l$  and  $s$  is  $\langle i, l \rangle$ -*expansionary*, i.e.,  $\min(\omega \setminus W_{g(i,l),s}) > \min(\omega \setminus W_{g(i,l),s-1})$ . Or
- $\mathcal{R} = N_j$  and  $\varphi_{j,s}(a_j(x, s))$  converges, where  $x$  is the greatest number in the domain of  $a_j(-, s)$ , and

$$(\forall n, m \leq x)[n T_s m \Leftrightarrow \varphi_{j,s}(a_j(n, s)) R_s \varphi_{j,s}(a_j(m, s))].$$

Or

- $\mathcal{R} = P_k$ , and  $\varphi_{k,s}$  converges on all  $z \leq y_k(s)$ , and either  $x_k(s) \not R_s y_k(s)$  and  $\varphi_{k,s}(x_k(s)) \not E_s \varphi_{k,s}(y_k(s))$ , and both  $\varphi_{k,s}(x_k(s))$  and  $\varphi_{k,s}(y_k(s))$  have been  $E$ -collapsed to elements restrained by higher priority strategies; or at least one of the values  $\varphi_{k,s}(x_k(s))$  and  $\varphi_{k,s}(y_k(s))$  has not as yet been  $E$ -collapsed to any element restrained by higher priority strategies, and  $x_k(s) \not R_s y_k(s)$  and  $\varphi_{k,s}(x_k(s)) \not E_s \varphi_{k,s}(y_k(s))$ ; or  $x_k(s) R_s y_k(s)$ .

At stage  $s > 0$  a number  $x$  is said to have been *created* by a requirement  $\mathcal{R}$ , if

- $\mathcal{R} = Q_l$ , and  $x$  is  $E_s$ -equivalent to some  $c_i$ , where  $\gamma_l(t) = \langle c_1, \dots, c_i, \dots, c_n \rangle$ , or to some  $a_l(x, t)$ , for some  $t \leq s$ ;
- $\mathcal{R} = N_j$ , and  $x$  is  $E_s$ -equivalent to some  $a_j(x, t)$ , for some  $t \leq s$ ;

a number is *new* at  $s$ , if it is bigger than all numbers (that are  $E_s$ -equivalent to numbers) so far mentioned in the construction.

We are now ready to give the construction.

*Step 0.* Initialize all requirements.

*Step  $s + 1$ .* Let  $\mathcal{R}$  be the least requirement that requires attention at stage  $s + 1$ . We say that  $\mathcal{R}$  *acts* at  $s + 1$ . Notice that there always exists such a requirement, as at each stage infinitely many requirements are initialized. We distinguish the following cases. (For simplicity, when describing the various parameters, or the various approximations to the equivalence relations, or to partial computable functions, we omit to mention the stage  $s$ : thus for instance,  $x \not\mathcal{R} y$  has to be read as  $x \not\mathcal{R}_s y$ , and so on.)

- (1) If  $\mathcal{R} = Q_l$ , then we take action as follows:
  - (a) if  $Q_l$  is initialized, then let  $n$  be the number of the distinct equivalence classes created by  $E$ , up to  $s$ , as the result of the actions taken by the higher priority requirements, and let  $\{b_1, \dots, b_n\}$  be representatives of these equivalence classes. Choose  $\gamma_l$  to be the least (by code)  $n$ -tuple of numbers that are currently pairwise non-equivalent in  $R$ ;
  - (b) if there exist  $1 \leq i, j \leq n$ ,  $i \neq j$ , such that  $c_i R c_j$  (where  $c_i$  and  $c_j$  are the  $i$ -th and  $j$ -th components, respectively, of  $\gamma_l$ ), then initialize  $Q_l$ ;
  - (c) if neither of the previous two cases holds then:
    - (i) take the least number  $x$  for which  $a_l(x)$  is not defined, and define  $a_l(x)$  to be a new number;
    - (ii) for every  $z \leq a_l(x)$  such that  $z \not\mathcal{E} b_i$ , every  $b_i$ , and  $z \not\mathcal{E} a_l(y)$ , every existing  $a_l(y)$ , then  $E$ -collapse  $z$  and  $d$ , where  $d := a_l(0)$ ;
    - (iii)  $E$ -collapse existing  $a_l(y)$  and  $a_l(z)$  if  $y R z$ ;
 put into  $\rho^{Q_l}(s + 1)$  all numbers  $b_i$ ,  $1 \leq i \leq n$ , and  $a_l(y)$ ,  $y \leq x$ , where  $x$  is the greatest number for which  $a_l(\cdot)$  is defined.
- (2) If  $\mathcal{R} = N_j$  then we act as follows:
  - (a) if  $N_j$  is initialized then we appoint new elements  $a_j(0)$  and  $a_j(1)$ ;
  - (b) otherwise, let  $x$  be the greatest number such that  $a_l(\cdot)$  is defined: for every  $y, z \leq x$ ,  $E$ -collapse all  $a_j(y), a_j(z)$  if  $y T z$ ; finally, appoint a new  $a_j(x + 1)$ ;
 put in  $\rho^{N_j}(s + 1)$  all numbers  $a_j(i)$ ,  $i \leq x$ , where  $x$  is the greatest number for which  $a_j(\cdot)$  is defined.
- (3) If  $\mathcal{R} = P_k$  then we act as follows:
  - (a) if  $P_k$  is initialized then appoint  $x_k$  and  $y_k$  so that  $\langle x_k, y_k \rangle$  is the least pair  $\langle x, y \rangle$  for which  $x < y$ ,  $x \not\mathcal{R} y$ , and  $\langle x, y \rangle > \langle x_k(t), y_k(t) \rangle$ , for every  $t \leq s$ ;
  - (b) if  $x_k \not\mathcal{R} y_k$  and  $\varphi_k(x_k) \not\mathcal{E} \varphi_k(y_k)$ , but both  $\varphi_k(x_k)$  and  $\varphi_k(y_k)$  are already  $E$ -equivalent to restrained elements, (i.e., elements belonging to the set

$$S(s) := \{z : (\exists \mathcal{R})[\mathcal{R} \text{ has higher priority than } P_k \ \& \ z \in \rho^{\mathcal{R}}(s)]\}$$

then initialize  $P_k$ ;

- (c) if  $x_k \not\mathcal{R} y_k$  and  $\varphi_k(x_k) \not\mathcal{E} \varphi_k(y_k)$ , and at least one of them has not as yet been  $E$ -collapsed to a restrained element, then  $E$ -collapse  $\varphi_k(x_k)$  and  $\varphi_k(y_k)$ ;
- (d) if  $x_k R y_k$ , then initialize  $P_k$ .

After acting, end the stage, and initialize all lower priority requirements.

**Verification.** We now check that the construction works.

**Lemma 2.2.** *If every higher priority requirement acts only finitely often, then  $P_k$  acts only finitely often.*

*Proof.* Assume that every higher-priority requirement acts only finitely often, and suppose, towards a contradiction, that  $P_k$  acts infinitely often. Let  $s$  be the last stage at which a higher-priority requirement acts. Let  $S$  be the set of all elements  $E$ -equivalent to some element in  $S(s)$  which is the finite set restrained by higher priority actions by stage  $s$ , as in (3b) of step  $s + 1$ . Thus  $S$  is the union of finitely many  $E$ -equivalence classes. For  $P_k$  to act infinitely often, we must have  $\varphi_k$  total, and by (3a) and (3d), we test all possible choices of  $x_k, y_k$ , with  $x_k < y_k$  and  $x_k \not R y_k$ : for each one of these pairs (by definition of  $P_k$  requiring attention) we have that  $\varphi_k(x_k), \varphi_k(y_k) \in S$  and  $\varphi_k(x_k) \not E \varphi_k(y_k)$ . But this would imply that there exists a 1-1 function from the infinitely many distinct equivalence classes of  $R$  to the finitely many equivalence classes in  $S$ . Therefore, we must have that either  $P_k$  eventually does not require attention because  $\varphi_k$  is not total; or we find  $x_k, y_k$  such that  $x_k \not R y_k$ ,  $\varphi_k(x_k), \varphi_k(y_k) \in S$ , and  $\varphi_k(x_k) E \varphi_k(y_k)$ ; or (3c) applies.  $\square$

**Lemma 2.3.** *If every higher priority requirement acts only finitely often, then  $N_j$  acts only finitely often.*

*Proof.* Suppose, towards a contradiction, that  $N_j$  acts infinitely often. Let  $s$  be the last stage at which a higher priority requirement acts, i.e.  $N_j$  is initialized for the last time at stage  $s$ . We consider the assignments of  $a_j(k)$  after stage  $s$ . Then for each  $n, m$ ,

$$n T m \Leftrightarrow \varphi_j(a_j(n)) R \varphi_j(a_j(m)).$$

Thus the function  $i \mapsto \varphi_j(a_j(i))$  gives a reduction of  $T$  to  $R$ . This yields a contradiction since  $R$  is non-universal, showing that  $N_j$  acts only finitely often.  $\square$

**Lemma 2.4.** *Suppose that  $W_{g(i,l)}$  is finite for each  $l$ . Then  $E$  is  $\leq_c$ -incomparable to  $R$ .*

*Proof.* By assumption, each  $Q_l$  acts only finitely often, so by Lemmas 2.2 and 2.3, every requirement acts only finitely often.

We now argue that since each requirement acts only finitely often, each succeeds. Since every requirement acts only finitely often, we can consider the final assignments of  $a_j(k)$  for the requirement  $N_j$ . Either  $\varphi_j$  is not total or for some  $a_j(n), a_j(m)$ ,

$$a_j(n) E a_j(m) \Leftrightarrow n T m,$$

but not

$$n T m \Leftrightarrow \varphi_j(a_j(n)) R \varphi_j(a_j(m)).$$

Thus  $\varphi_j$  is not a reduction of  $E$  to  $R$ , and  $N_j$  is satisfied. Since  $P_k$  acts only finitely often and  $R$  has infinitely many classes, either  $\varphi_k$  is not total or there are  $x_k, y_k$  so that  $x_k \not R y_k$  but  $\varphi_k(x_k) E \varphi_k(y_k)$ . Thus  $\varphi_k$  is not a reduction of  $R$  to  $E$ , and  $P_k$  is satisfied.  $\square$

**Lemma 2.5.** *Suppose that for some  $l$ ,  $W_{g(i,l)} = \omega$ . Then  $E \equiv_c R$ .*

*Proof.* Let  $Q_l$  be of highest priority so that  $W_{g(i,l)} = \omega$ . By Lemmas 2.2 and 2.3, every higher priority requirement acts only finitely often. Consider the least stage  $t$  at which every higher priority action stops acting, giving  $n$  distinct equivalence classes. Further, consider a stage  $s > t$  where  $Q_l$  has found (through (1a) and (1d)) the appropriate choice of  $n$   $R$ -non-equivalent elements, thus choosing the final  $\gamma_l$ . After this stage  $s$ , every time  $Q_l$  picks a number  $a_l(x)$ , then this is the final value of  $a_l(x, s)$ , and  $Q_l$  creates a class  $[d]$ , with  $d = a_l(0)$ , which contains all elements

previously created for all lower priority requirements, and it will also contain all elements later created for lower priority requirements (when  $Q_l$  acts again, it will  $E$ -collapse them to  $d$ ). We now provide reductions witnessing that  $E \equiv_c R$ .

To see  $E \leq_c R$ , consider the function  $f$  constructed as follows: We begin with the finitely many elements created for higher priority requirements, which are grouped into finitely many finite  $E$ -equivalence classes, as created at stage  $s$ :  $[b_1]_E = [b_1]_{E_s}, \dots, [b_n]_E = [b_n]_{E_s}$ . We have found  $c_1, \dots, c_n$  so that  $b_i E b_j$  if and only if  $c_i R c_j$ , so that the assignment  $b_k \mapsto c_k$  satisfies

$$(\forall 1 \leq h, k \leq n)[b_h E b_k \Leftrightarrow c_h R c_k].$$

Let  $T$  be the set of elements  $a \in \omega$  created on a  $Q_l$ -stage after  $s$  (i.e., a stage  $\geq s$  where  $\min(\omega \setminus W_{g(i,l)})$  increases). Note that  $d \in T$ . For any  $a \in T$ ,  $a$  is created to copy  $R$  on some number, i.e.  $a = a_k(x)$  for some  $x$ . So, consider the function  $f$ ,

$$f(x) = \begin{cases} c_i, & \text{if } x \in [b_i]_{E_s}, \text{ some } 1 \leq i \leq n, \\ y, & \text{if } x \in T, \text{ say } x = a_k(y), \\ 0, & \text{otherwise.} \end{cases}$$

(Notice that  $0 = f(a_k(0)) = f(d)$ .) The numbers  $x$  not created on  $Q_l$ -stages, are either in some  $[b_i]_{E_s}$ , or are created for lower priority requirements: in this latter case,  $x E d$ , for which we have defined  $f(x) = f(d)$ . This function  $f$  is computable and witnesses that  $E \leq_c R$ .

For the converse, the mapping  $x \mapsto a_k(x)$  provides a reduction from  $R$  to  $E$ . □

This concludes the proof of the theorem. □

**Corollary 2.6.** *The following hold:*

- (1) *If  $R$  is any ceer with infinitely many classes, then  $\{i \mid R_i \equiv_c R\}$  is  $\Sigma_3^0$  complete.*
- (2) *If  $R$  is any ceer with infinitely many classes, then  $\{i \mid R_i \geq_c R\}$  is  $\Sigma_3^0$  complete.*
- (3) *If  $R$  is any non-universal ceer with infinitely many classes, then  $\{i \mid R_i \leq_c R\}$  is  $\Sigma_3^0$  complete.*
- (4) *If  $R$  is universal, then  $\{i \mid R_i \leq_c R\} = \omega$ , thus is decidable.*
- (5) *If  $R$  has only finitely many classes, then  $\{i \mid R_i \leq_c R\}$  is  $\Pi_2^0$  complete.*
- (6) *If  $R$  has finitely many, but at least 2, classes, then  $\{i \mid R_i \geq_c R\}$  is  $\Sigma_2^0$  complete.*
- (7) *If  $R$  has only one class, then  $\{i \mid R_i \geq_c R\} = \omega$ , thus is decidable.*
- (8) *If  $R$  has finitely many, but at least 2, classes, then  $\{i \mid R_i \equiv_c R\}$  is  $d\text{-}\Sigma_2^0$  complete.*

*Proof.* It is straightforward to check that the proposed sets lie in the appropriate level of the arithmetical hierarchy. To how hardness, we prove the items one by one.

- (1) There are two cases. If  $R$  is universal, this is exactly Theorem 5.1 in [1]. If  $R$  is non-universal, this follows directly from the previous theorem.
- (2) If  $R$  is universal, then the claim follows from (1). If not, then it follows from the previous theorem.
- (3) This follows from the previous theorem.
- (4) Trivial.

- (5) Note that if  $R$  has  $k$  classes, then  $E \leq R$  if and only if  $E$  has  $\leq k$  classes. It is easy to show that having  $\leq k$  classes is a  $\Pi_2^0$  complete property:  $E$  has  $\leq k$  classes if and only if

$$(\forall x_0, \dots, x_k)(\exists i, j \leq k)[i \neq j \ \& \ x_i \ E \ x_j].$$

Let us now show that this property is  $\Pi_2^0$  hard. It is known that  $\text{Inf} = \{i \mid W_i \text{ infinite}\}$  is  $\Pi_2^0$  complete: it is easy to see that there is a computable function  $f$  such that, for every  $i$ ,  $E_{f(i)}$  is a ceer satisfying:

$$\begin{aligned} i \in \text{Inf} &\Rightarrow [0]_{E_{f(i)}} = \omega, \\ i \notin \text{Inf} &\Rightarrow (\exists x)(\forall y \geq x) \left[ [y]_{E_{f(i)}} = \{y\} \right]. \end{aligned}$$

- (6) If  $R$  has  $k \geq 2$  classes, then  $E \geq_c R$  holds if and only if  $E \leq_c S$ , where  $S$  has  $k - 1$  classes. Thus, by (5), this is  $\Sigma_2^0$  complete.
- (7) Trivial.
- (8) By combining the arguments in (5) and (6). Note that if  $R$  has exactly one class, then  $\{i \mid R_i \equiv_c R\} = \{i \mid R_i \leq_c R\}$  is  $\Pi_2^0$  complete by (5).

□

### 3. THE INDEX SET OF THE EFFECTIVELY INSEPARABLE CEERS

A pair of disjoint sets  $A, B$  is *effectively inseparable* (shortly, *e.i.*) if there exists a partial computable function  $\psi$  (called a *productive function* for the pair) such that, for every pair of c.e. indices  $u, v$ ,

$$A \subseteq W_u \ \& \ B \subseteq W_v \ \& \ W_u \cap W_v = \emptyset \Rightarrow \psi(u, v) \downarrow \ \& \ \psi(u, v) \notin W_u \cup W_v.$$

It is not difficult to see:

**Lemma 3.1.** *Every e.i. pair of c.e. sets, has a total productive function.*

*Proof.* The proof is similar to the one showing that every productive set has a total productive function, see e.g. [19, p. 41]. □

A ceer  $R$  is called *effectively inseparable* (shortly, *e.i.*), see [1], if every pair of distinct equivalent classes  $[a]_R, [b]_R$  is e.i.. If indices for productive functions for the various pairs of equivalence classes can be found uniformly (i.e., there exists a computable function  $g$  such that, for every pair  $a, b$ , if  $a \not\equiv b$  then  $\varphi_{g(a,b)}$  is a productive function for the pair  $[a]_R, [b]_R$ ), then  $R$  is said to be *uniformly effectively inseparable* (or, shortly, *u.e.i.*), [1]. It is proved in [1] that the index set of the u.e.i. ceers is  $\Sigma_3^0$  complete, and is posed as an open question whether the index set of the e.i. ceers is  $\Pi_4^0$  complete. In the following theorem we answer this question.

**Theorem 3.2.** *The index set of the e.i. ceers is  $\Pi_4^0$  complete.*

*Proof.* It is straightforward to check that the index set of the e.i. ceers is  $\Pi_4^0$ . Now, every  $\Pi_4^0$  set  $S$  can be described as  $S = \{i : (\forall j)[W_{g(i,j)} \text{ is cofinite}]\}$ : this is an easy consequence of the fact that the index set  $\{i : W_i \text{ is cofinite}\}$  is  $\Sigma_3^0$  complete (see e.g., [19, p. 66]). Therefore, we can fix a recursive function  $g(i, j)$  so that  $S := \{i \mid (\forall j)[W_{g(i,j)} \text{ is cofinite}]\}$  is a  $\Pi_4^0$  complete set. We now produce a function which, on input  $i$ , uniformly produces a ceer  $E$  so that  $E$  is e.i. if and only if  $i \in S$ . In what follows, we describe the enumeration of  $E$  for a given  $i$ .



Given a set  $X$  let  $X^{[2]}$  denote the collections of all subsets of  $X$  consisting of exactly two elements. We fix a pair of recursive bijections  $m : \omega^{[2]} \rightarrow \omega$  and  $n_0 : (2\omega)^{[2]} \rightarrow \omega$ . We then define  $n : \omega^{[2]} \rightarrow \omega$  so that  $n(x) = n_0(x)$  if  $x \in (2\omega)^{[2]}$ , and  $n(x) = m(x)$  otherwise.

**Requirements and strategies.** We have the following requirements, where  $a < b$ , i.e.,  $\{a, b\} \in \omega^{[2]}$ .

$$P_j^{a,b} : [j, \infty) \subseteq W_{g(i,m(a,b))} \cap W_{g(i,n(a,b))} \ \& \ a \not E b \Rightarrow f_j^{a,b} \text{ is a productive function for } [a]_E, [b]_E,$$

$$N_j^{a,b} : [j, \infty) \not\subseteq W_{g(i,m(a,b))} \cap W_{g(i,n(a,b))} \ \& \ a, b \in (2\omega)^{[2]} \Rightarrow \varphi_j \text{ is not productive for } [a]_E, [b]_E.$$

The requirements are partitioned, in the obvious way, into *P-requirements* and *N-requirements*.

**Remark 3.3.** If  $i \in S$  then for every  $r$ ,  $W_{g(i,r)}$  is cofinite, and thus for every  $a \neq b$  there is  $j_{a,b}$  such that  $[j_{a,b}, \infty) \subseteq W_{g(i,m(a,b))} \cap W_{g(i,n(a,b))}$ : hence if  $a \not E b$ , and we satisfy  $P_{j_{a,b}}^{a',b'}$ , where  $a'$  and  $b'$  are the least elements in the  $E$ -equivalence classes of  $a$  and  $b$ , respectively, then we guarantee that  $f_{j_{a,b}}^{a',b'}$  is a productive function for the pair  $[a]_E, [b]_E$ .

Viceversa, if  $i \notin S$ , then there is  $r$  such that  $W_{g(i,r)}$  is not cofinite, nor is any  $W_{g(i,r')} \cap W_{g(i,r)}$ , and thus if  $a, b \in 2\omega$ ,  $a \neq b$ , are such that  $n(a, b) = r$ , then for every  $j$ , one has  $[j, \infty) \not\subseteq W_{g(i,m(a,b))} \cap W_{g(i,n(a,b))}$ . In this case, if we satisfy all  $N_j^{a,b}$ -requirements, then we guarantee that the pair  $[a]_E, [b]_E$  is not effectively inseparable.

We will never cause non-equal even elements  $a, b$  to become  $E$ -equivalent, and in fact each even number will be the least element in its equivalence class.  $N_j^{a,b}$ -requirements will only pose restraints asking that two elements not become equivalent, but will never cause  $E$ -collapse.

We fix a priority ordering in which if  $j < j'$  then  $P_j^{a,b} < P_{j'}^{a,b}$ .

We first describe the actions of each requirement separately. The reader should think of  $a, b$  as the least numbers in their respective equivalence classes, and  $a \neq b$ .

*P-Requirements.* A  $P_j^{a,b}$ -requirement performs the standard effective inseparability strategy: it builds a computable function  $f = f_{a,b}^j$  as follows. For the least (by code) pair  $u, v$ , on which  $f(u, v)$  is still undefined, define  $f(u, v)$  to be an odd number  $y$  larger than any number considered so far: if  $y$  is observed to be enumerated into  $W_u$ , cause  $y E b$ ; if  $y$  is observed to be enumerated into  $W_v$ , then cause  $y E a$ . The strategy for  $P_j^{a,b}$  acts every time the least element of  $[j, \infty) \setminus W_{g(i,m(a,b))} \cap W_{g(i,n(a,b))}$  enters  $W_{g(i,m(a,b))} \cap W_{g(i,n(a,b))}$ , i.e. when there is evidence that eventually  $[j, \infty) \subseteq W_{g(i,m(a,b))} \cap W_{g(i,n(a,b))}$ : we say that in this case the strategy  $P_j^{a,b}$  takes the *infinite outcome*; otherwise  $P_j^{a,b}$  takes the *finite outcome*. It is clear that either  $P_j^{a,b}$  takes the infinite outcome infinitely many times (we say that in this case that  $P_j^{a,b}$  has outcome  $\infty$ ), or from some point on,  $P_j^{a,b}$  always takes the finite outcome (we say that in this case that  $P_j^{a,b}$  has outcome  $f$ ).

We summarize as follows:

$$\begin{aligned} \text{outcome } \infty : [j, \infty) &\subseteq W_{g(i,m(a,b))} \cap W_{g(i,n(a,b))}; \\ \text{outcome } f : [j, \infty) &\not\subseteq W_{g(i,m(a,b))} \cap W_{g(i,n(a,b))}. \end{aligned}$$

*N-Requirements.* An  $N_j^{a,b}$ -requirement acts only if, and immediately after,  $P_j^{a,b}$  has taken the finite outcome, and its action is as follows: Choose a pair  $u, v$  so that we (via the Recursion Theorem) control the enumeration of  $W_u$  and  $W_v$ . Let  $W_u$  enumerate  $[a]_E$  and  $W_v$  enumerate  $[b]_E$ , and wait for a stage when  $\varphi_j(u, v)$  converges to a value, say  $y$ . If  $y \in [a]_E \cup [b]_E$ , then the requirement does nothing further. Otherwise, we distinguish:

Case 1:  $y$  is an odd number chosen as  $f_{j'}^{a',b'}(u', v')$  for some  $j', a', b'$  with  $a', b'$  least numbers in their equivalence classes, and  $a' \neq a$  (if  $a' = a$  and  $b' \neq b$ , the requirement acts symmetrically). In this case, we enumerate  $y$  into  $W_v$ . We place a restraint for  $y$  to never enter  $[a]$ .

Case 2: Not case 1. In this case, we enumerate  $y$  into  $W_u$  and place a restraint for  $y$  to never enter  $[b]$ .

Every time the least element of  $[j, \infty) \cap W_{g(i,m(a,b))} \cap W_{g(i,n(a,b))}$  enters  $W_{g(i,m(a,b))} \cap W_{g(i,n(a,b))}$ ,  $N_j^{a,b}$  will be injured, so if  $[j, \infty) \subseteq W_{g(i,m(a,b))} \cap W_{g(i,n(a,b))}$ , then  $N$  will not prevent effective inseparability of the pair  $[a], [b]$ .

*The Recursion Theorem.* In carrying on the strategies for the  $N$ -requirements, we use indices that we control by the Recursion Theorem, or, more precisely, we make use of a computable sequence of fixed points. Since a computable sequence of indices can be viewed as the range of a computable function  $f$ , a formal justification to this argument is provided by the Case Functional Recursion Theorem, see [3]: see also [16] for useful comments about this theorem.

**Lemma 3.4** (Case Functional Recursion Theorem). *Given a partial computable functional  $F$ , there is a total computable function  $f$  such that, for every  $e, x$ ,*

$$F(f, e, x) = \varphi_{f(e)}(x).$$

*The tree of strategies.* We organize the construction on a tree  $T$ , which is a set of strings on the alphabet  $\{g, \infty, f\}$ . With respect to the above discussion of requirements and their outcomes, it is convenient to use also an additional outcome  $g$ , which for a requirement  $P_j^{a,b}$  or  $N_j^{a,b}$ , will record the fact that at least one among  $a, b$  is not the least number in its equivalence class.

The tree  $T$ , and the function

$$\mathcal{R} : T \longrightarrow \text{Requirements},$$

assigning requirements to the nodes of  $T$ , are defined as follows, where  $\lambda$  denotes the empty string.

**Definition 3.5.**  $\lambda \in T$ , and  $\mathcal{R}(\lambda)$  is the highest priority  $P$ -requirement.

- If  $\sigma \in T$ , and  $\mathcal{R}(\sigma) = P_j^{a,b}$  is a  $P$ -requirement, then  $\sigma \hat{\langle} o \rangle \in T$ , for  $o \in \{g, \infty, f\}$ :
  - all requirements  $P_{j'}^{a,b}$  for  $j' > j$  are declared to be *cancelled* by  $\sigma \hat{\langle} o \rangle$ . (Since if  $[j, \infty) \subseteq W_{g(i,m(a,b))} \cap W_{g(i,n(a,b))}$ , then  $[j', \infty) \subseteq W_{g(i,m(a,b))} \cap W_{g(i,n(a,b))}$  for all  $j' \geq j$ , thus the requirement  $P_{j'}^{a,b}$  need not be considered again below  $\sigma \hat{\langle} o \rangle$ .)  $\mathcal{R}(\sigma \hat{\langle} o \rangle)$  is the highest priority  $P$ -requirement not assigned to any  $\tau \subseteq \sigma$  and not cancelled by any  $\tau \subseteq \sigma \hat{\langle} o \rangle$ .

- If  $a, b$  are both even, then  $\mathcal{R}(\sigma \hat{\langle f \rangle}) = N_j^{a,b}$ ; otherwise  $\mathcal{R}(\sigma \hat{\langle f \rangle})$  is the highest priority  $P$ -requirement not assigned to any  $\tau \subseteq \sigma$  and not cancelled by any  $\tau \subseteq \sigma$ .
- $\mathcal{R}(\sigma \hat{\langle g \rangle})$  is the highest priority  $P$ -requirement not assigned to any  $\tau \subseteq \sigma$  and not cancelled by any  $\tau \subseteq \sigma$ .
- If  $\sigma \in T$ , and  $\mathcal{R}(\sigma)$  is an  $N$ -requirement, then  $\sigma \hat{\langle f \rangle} \in T$  (by construction,  $a, b$  will be the least elements in their respective equivalence classes, so we do not consider the  $g$  outcome);  $\mathcal{R}(\sigma \hat{\langle f \rangle})$  is the highest priority  $P$ -requirement not assigned to any  $\tau \subseteq \sigma$  and not cancelled by any  $\tau \subseteq \sigma$ .
- No other string on  $\{\infty, f, g\}$  lies in  $T$ .

The elements of  $T$  are ordered by the *lexicographical order*  $\leq$ , generated by the ordering on the alphabet, for which  $g < \infty < f$ : thus  $\sigma \leq \tau$  if  $\sigma \subseteq \tau$  or, for the least  $i$  such that  $\sigma, \tau$  are both defined on  $i$ , and  $\sigma(i) \neq \tau(i)$ , we have that  $\sigma(i) < \tau(i)$ : in this latter case we also write  $\sigma <_L \tau$ .

*The environments of the strategies.* Notice that the function  $\mathcal{R}$ , assigning requirements to nodes, is computable. For every  $\sigma$ , we also call  $\mathcal{R}(\sigma)$  a *strategy*. Each strategy has several parameters: if  $\mathcal{R}(\sigma) = P_j^{a,b}$  then it uses the parameter  $f_{\sigma,s}$  (approximating the function  $f_j^{a,b}$  of the above informal description), whereas if  $\mathcal{R}(\sigma)$  is an  $N$ -requirement, then it uses the parameters  $u_\sigma(s), v_\sigma(s)$ , and  $y_\sigma(s)$  (approximating  $u, v, y$  of the above informal description).

**The construction.** At stage  $s$  we define a finite string  $\delta_s$  of length  $|\delta_s| \leq s$ , which approximates the true path at stage  $s$ . The string  $\delta_s$  is defined by substages: at substage  $n$ , we define  $\sigma_n = \delta_s \upharpoonright n$ . A number is *new* at any substage of stage  $s > 0$  if it is bigger than all numbers already  $E$ -collapsed to numbers so far mentioned in the construction. If  $\mathcal{R}(\sigma) = P_{a,b}^j$  is a  $P$ -strategy, then a stage  $s$  is  $\sigma$ -*expansive* if for no  $t < s$  did we have  $\sigma \subseteq \delta_s$ , or  $\min([j, \infty) \setminus (W_{g(i,m(a,b),s)} \cap W_{g(i,n(a,b),s)}))$  has increased since the last stage  $t < s$  which was  $\sigma$ -*true*, i.e., at which  $\sigma \subseteq \delta_t$ . A number  $z$  is *created by*  $\mathcal{R}(\sigma)$  at  $s$ , if  $y$  is in the range of  $f_{\sigma,s}$ ; or,  $z$  is appointed as  $u_\sigma(s)$  or  $v_\sigma(s)$ , or  $y_\sigma(s)$ . At stage  $s$ , we *initialize a strategy*  $\mathcal{R}(\sigma)$  if we set  $f_{\sigma,s} = \emptyset$  and we set  $u_\sigma(s), v_\sigma(s)$ , and  $y_\sigma(s)$  to be undefined. If  $y$  has been created by  $\mathcal{R}(\sigma) = P_j^{a,b}$ , by stage  $s$ , then  $y$  is *active* at  $s$  if  $\mathcal{R}(\sigma)$  has not been initialized after  $y$  has been created, and  $\mathcal{R}(\sigma)$  has not as yet  $E$ -collapsed  $y$  to either  $a$  or  $b$ .

*Stage 0.* Initialize all strategies  $\mathcal{R}(\sigma)$ .

*Stage  $s + 1$ .* Proceed according to the following substages (as in the proof of Theorem 2.1, when describing the various parameters, or the various approximations to c.e. sets, partial computable functions, or  $E$ , we omit mentioning the stage  $s$ ):

*Substage 0.* Let  $\delta_{s+1} \upharpoonright 0 = \lambda$ .

*Substage  $n + 1$ .* If  $n = s$  then go to next stage. Otherwise, take the first relevant case that applies below:

- (1) Suppose that  $\mathcal{R}(\sigma_n) = P_j^{a,b}$ .
  - (a) If one among  $a, b$  is not the least element of its  $E$ -equivalence class, then let  $\sigma_{n+1} = \sigma_n \hat{\langle g \rangle}$ .

- (b) If  $s$  is a  $\sigma_n$ -expansionary stage, let  $\sigma_{n+1} = \sigma_n \hat{\langle \infty \rangle}$ . Then extend  $f_{\sigma_n}$  by considering the least (by code) pair  $(u, v)$  on which  $f_{\sigma_n}$  is not defined, and define  $f_{\sigma_n}(u, v) = y$ , for some new odd  $y > a, b$ . Also, if  $f_{\sigma_n}(u', v') = y'$  has been already defined, and up to now  $y'$  has been active, but currently  $y' \in W_{u'} \cup W_{v'}$ , then
- (i) if  $y' \in W_{u'}$  then  $E$ -collapse  $y'$  and  $b$ ;
  - (ii) if  $y' \in W_{v'}$  then  $E$ -collapse  $y'$  and  $a$ .
- (c) Otherwise, let  $\sigma_{n+1} = \sigma_n \hat{\langle f \rangle}$ .
- (2) If  $\mathcal{R}(\sigma_n) = N_j^{a,b}$ , then let  $\sigma_{n+1} = \sigma_n \hat{\langle f \rangle}$ . We act according to the first applicable case among the following:
- (a)  $\mathcal{R}(\sigma_n)$  is initialized: assume by the Recursion Theorem that  $u$  and  $v$  are indices that we control, such that  $u$  and  $v$  are new numbers; let  $u_\sigma(s+1) = u$ ,  $v_\sigma(s+1) = v$ ;
  - (b)  $\varphi_j(u, v)$  converges to some number  $y$  (where  $u = u_\sigma(s)$ ,  $v = v_\sigma(s)$ , and we define  $y_\sigma(s+1) = y$ );
    - (i) if  $s+1$  is the first  $\sigma_n$ -true stage at which  $\varphi_j(u, v)$  converges, then end the stage (thus initializing all strategies of lower priority);
    - (ii) if  $y$  is  $E$ -equivalent to some active  $f_\tau(u', v')$  created by  $\mathcal{R}(\tau) = P_{j'}^{a', b'}$ , with  $\tau \hat{\langle \infty \rangle} \subseteq \sigma_n$ , and  $\{a', b'\} \neq \{a, b\}$ , then if  $a' \neq a$ , enumerate  $y$  into  $W_v$ ; otherwise (i.e.,  $a' = a$ , but  $b' \neq b$ ), enumerate  $y$  into  $W_u$ ; (notice that by the way requirements are assigned to strings in  $T$ , there is no  $\tau \hat{\langle \infty \rangle} \subseteq \sigma_n$  with  $\mathcal{R}(\tau) = P_{j'}^{a, b}$ , any  $j'$ ); Also, enumerate  $[a]_E$  into  $W_u$  and  $[b]_E$  into  $W_v$ .
    - (iii) if  $y \in [a]_E \cup [b]_E$  then enumerate  $[a]_E$  into  $W_u$ , and enumerate  $[b]_E$  into  $W_v$ ;
    - (iv) otherwise, enumerate  $[a]_E \cup \{y\}$  in  $W_u$ , and  $[b]_E$  in  $W_v$ .

At the end of the stage, initialize all strategies  $\mathcal{R}(\tau)$ , with  $\tau \geq \delta_{s+1}$ . Define  $E_{s+1}$  to be the least equivalence relation generated by  $E_s$  plus the pairs  $E$ -collapsed at stage  $s+1$ . This ends Stage  $s+1$ .

Finally, let

$$E = E_i = \bigcup_s E_s.$$

**The verification.** The following holds:

**Lemma 3.6.** *There exists an infinite path  $\text{tp} \upharpoonright n$  through the tree  $T$  such that, for every  $n$ ,*

$$\text{tp} \upharpoonright n = \liminf_s \delta_s \upharpoonright n,$$

(where the  $\liminf$  is taken with respect to the lexicographical order of strings of  $T$ ), and  $\text{tp} \upharpoonright n$  eventually does not end the stage.

*Proof.* The proof is by induction on  $n$ . Suppose that the claim is true of  $n$ , and let  $s_0$  be the least stage such that there is no  $\sigma$ -true stage  $s \geq s_0$  for any  $\sigma <_L \text{tp} \upharpoonright n$ , and  $\text{tp} \upharpoonright n$  does not end the stage at  $s$ : thus  $s_0 > n$ . If there is a stage  $s_1 \geq s_0$  such that  $\text{tp} \upharpoonright n \hat{\langle g \rangle} \subseteq \delta_{s_1}$ , then for every  $\text{tp} \upharpoonright n$ -true  $s \geq s_1$  we have  $\text{tp} \upharpoonright n \hat{\langle g \rangle} \subseteq \delta_s$ , and if  $s > n+1$  then  $\text{tp} \upharpoonright n \hat{\langle g \rangle}$  does not end the stage, and clearly  $\text{tp} \upharpoonright n+1 = \text{tp} \upharpoonright n \hat{\langle g \rangle}$ . If for almost all true  $\text{tp} \upharpoonright n$ -true stages  $s \geq s_0$  we have  $\text{tp} \upharpoonright n \hat{\langle f \rangle} \subseteq \delta_s$ , then  $\text{tp} \upharpoonright n+1 = \text{tp} \upharpoonright n \hat{\langle f \rangle}$ , and  $\text{tp} \upharpoonright n+1$  ends at most twice, at any such  $s$ : namely, if  $s = n+1$ , and when we act through (2bi) of the construction. Otherwise there exist infinitely many true  $\text{tp} \upharpoonright n$ -true stages  $s \geq s_0$  at which  $\text{tp} \upharpoonright n \hat{\langle \infty \rangle} \subseteq \delta_s$ : thus  $\text{tp} \upharpoonright n+1 = \text{tp} \upharpoonright n \hat{\langle \infty \rangle}$  and  $\text{tp} \upharpoonright n+1$  does not end the stage at any such  $s > n+1$ .  $\square$

**Lemma 3.7.** *Let  $\sigma$  be so that  $\sigma \subset \text{tp}$  and  $\sigma \widehat{\langle} g \rangle \not\subset \text{tp}$ . If  $\sigma$  is an  $N_j^{a,b}$  or  $P_j^{a,b}$  strategy, then  $a$  and  $b$  are the least numbers in their respective equivalence classes.*

*Proof.* Immediate. □

**Lemma 3.8.** *At every stage  $s$ , in any equivalence class  $[c]_{E_s}$  there is at most one element which is even or active. If, at some stage  $s$  where  $c$  is not new, the class  $[c]_{E_s}$  contains no even or active element, then for all  $t > s$ ,  $[c]_{E_t}$  contains no even or active element. Similarly, if at some stage  $s$  where  $c$  is not new,  $[c]_{E_s}$  contains no element active for requirement  $P_j^{a,b}$ , then at no stage does  $[c]_E$  contain an element active for requirement  $P_j^{a,b}$ .*

*Proof.* We prove the first claim by induction. This is clearly true at stage 0 where every equivalence class has size 1. When we activate a new number, we choose it to be a new odd element, thus is inequivalent to any even or active number. When we collapse classes  $[a]$  and  $[y]$ , it is because some element  $y'$  in  $[y]$  is active and equals  $f_\sigma(a', b')$  for some  $a' \in [a]$  and some  $b'$  (or symmetrically, it equals  $f_\sigma(c', a')$  for some  $a' \in [a]$  and some  $c'$ ). We then make  $y'$  inactive and collapse  $[y]$  to  $[a]$ . Thus there is still at most one even or active element in the class  $[a]$ . The second statement is proved analogously: Any element which becomes active is new, thus is not  $E$ -equivalent to  $c$ , and the property of not containing an even or active element is preserved when a second class collapses with  $[c]$ . The last statement is similar. □

**Lemma 3.9.** *Every even number is the least number in its  $E$ -equivalence class.*

*Proof.* By the previous lemma, no two even numbers are ever equivalent.

We now show that if  $a$  is even, then  $a$  is the least number in its equivalence class. By the previous conclusion, it is enough to show that for every  $s$ , and odd number  $y$ , if  $y$  is  $E$ -collapsed to  $a$  at  $s$ , then  $y > a$ . Assume that the claim is true of all odd numbers  $y'$  already  $E$ -collapsed to  $a$  at stages  $s' < s$ . An odd number  $y$  can be moved to  $[a]_{E_s}$  at  $s$ , either because (1bi) or (1bii) for some  $P_j^{a,b}$ , but then  $y > a$ , by choice of  $y > a, b$  in (2); or  $y$  is  $E$ -collapsed, through (1bi) or (1bii) for some  $P_{j'}^{y', b'}$ , to some odd number  $y'$  previously  $E$ -collapsed to  $a$ , but then by induction and, again, choice of  $y$  by (1b) of the construction, we have  $y > y' > a$ . □

**Lemma 3.10.** *If  $i \in S$  then for every  $a, b$ , if  $a \not\mathcal{E} b$ , the pair  $[a]_E, [b]_E$  is e.i.. On the other hand, if  $i \notin S$  then there are  $a, b$  even numbers such that  $W_{g(i, n(a, b))}$  is co-infinite, and the pair  $[a]_E, [b]_E$  is not e.i.*

*Proof.* If  $i \in S$ , then (see Remark 3.3) for every  $a, b$  there exists a minimal  $j$ , such that  $[j, \infty) \subseteq W_{g(i, m(a, b))} \cap W_{g(i, n(a, b))}$ . Now, if  $a \not\mathcal{E} b$ , and  $a, b$  are the least numbers in their respective equivalence classes, then there exists  $n$  such that  $\mathcal{R}(\text{tp} \upharpoonright n) = P_j^{a, b}$  and  $\text{tp} \upharpoonright (n+1) = \text{tp} \upharpoonright n \widehat{\langle} \infty \rangle$ . (Notice that, under these assumptions, for every  $j' < j$  there is a node  $\tau_{j'}$  such that  $\mathcal{R}(\tau_{j'}) = P_{j'}^{a, b}$ , and  $\tau_{j'} \widehat{\langle} f \rangle \subset \text{tp}$ , and for every  $j' > j$  there is no node  $\tau \subset \text{tp}$  such that  $\mathcal{R}(\tau) = P_{j'}^{a, b}$ .) It is clear by the construction that  $f_{\text{tp} \upharpoonright n}$  is a computable function witnessing that the pair  $[a]_E, [b]_E$  is e.i.. Thus every pair of distinct  $E$ -equivalence classes is e.i., as on the true path the corresponding requirement relative to the least numbers in the classes, is satisfied.

Assume now that  $i \notin S$ . Then, by surjectivity of the function  $n_0$ , there exists a pair  $a, b$  of distinct even numbers such that, for every  $j$ ,  $[j, \infty) \not\subseteq W_{g(i, m(a, b))} \cap W_{g(i, n(a, b))}$ . By Lemma 3.9, for every

$j$  there is a (unique) node  $\tau_j \subset \text{tp}$  such that  $\mathcal{R}(\tau_j) = P_j^{a,b}$  and  $\tau_j \hat{\langle f \rangle} \subset \text{tp}$ . We show that, for every  $j$ ,  $\varphi_j$  can not be a total productive function for the disjoint pair  $[a]_E, [b]_E$ . Let  $s_0$  be the least stage such that there is no  $\tau$ -true stage  $s \geq s_0$  for any  $\tau <_L \tau_j \hat{\langle f \rangle}$ , and no  $\tau \subseteq \tau_j$  ends the stage after  $s_0$ . At the least  $\tau_j \hat{\langle f \rangle}$ -stage following  $s_0$  we appoint the last choice of  $u = u_{\tau_j \hat{\langle f \rangle}}(s)$ , and  $v = v_{\tau_j \hat{\langle f \rangle}}(s)$ . If we do not find  $y$  as in (2b) of the construction, then  $\varphi_j$  is not total. So assume that  $\varphi_j(u, v)$  converges to  $y$ , which is the final value of  $y_{\tau_j \hat{\langle f \rangle}}(s)$ . We claim that  $[a]_E \subseteq W_u$ ,  $[b]_E \subseteq W_v$ ,  $W_u \cap W_v = \emptyset$ , but  $y \in W_u \cup W_v$ , which implies that  $\varphi_j$  is not a productive function. Now, it is clear that  $[a]_E \subseteq W_u$ ,  $[b]_E \subseteq W_v$ , since there are infinitely many stages  $s$  at which we enumerate  $[a]_{E_s}$  into  $W_u$  and  $[b]_{E_s}$  into  $W_v$ . It is also clear that  $y \in W_u \cup W_v$ . It remains to see that  $W_u \cap W_v = \emptyset$ . Assume that  $\mathcal{R}(\tau_j \hat{\langle f \rangle})$  enumerates  $y$  into  $W_u$ : the case in which  $\mathcal{R}(\tau_j \hat{\langle f \rangle})$  enumerates  $y$  into  $W_v$  is similar.

By initialization in (2a) and Lemma 3.8, the number  $y$  will never be equivalent to an element active for a  $\tau > \tau_j \hat{\langle f \rangle}$ .

For  $y$  to eventually become  $E$ -equivalent to  $a$  or  $b$ , it must be equivalent at stage  $s_0$  to some active element  $d$  for some  $\mathcal{R}(\tau) = P_{j'}^{a',b'}$  with  $\tau \hat{\langle \infty \rangle} \subseteq \tau_j$ . By our use of the outcome  $g$ ,  $a', b'$  are the least numbers in their equivalence classes (and so are  $a$  and  $b$ ), and since there is no such  $\tau$  with  $\tau \hat{\langle \infty \rangle} \subseteq \tau_j$  and  $\mathcal{R}(\tau) = P_{j'}^{a,b}$ , any  $j'$ , we may conclude that  $\{a', b'\} \neq \{a, b\}$ . If  $a' \neq a$ , then  $\mathcal{R}(\tau_j \hat{\langle f \rangle})$  enumerates  $y \in W_v$ , contrary to assumption. Therefore  $a = a'$ : we can exclude the subcase  $b = b'$ , because otherwise  $P_{a,b}^j$  would be cancelled along the true path, by the way requirements are assigned to nodes of the tree, and the fact that in this case we would have  $j' < j$ . Thus we are left to consider the case  $a = a'$  and  $b \neq b'$ . If  $d$  remains active at all future stages, then  $y$  cannot be equivalent to any even number by Lemma 3.8. Otherwise,  $y$  collapses with  $a'$  or  $b'$ . In either case, it cannot in the future collapse with  $b$ , since all three of  $a', b', b$  are the least elements of their equivalence classes and  $b \notin \{a', b'\}$ .  $\square$

This concludes the proof of the theorem.  $\square$

It is proved in [1] that the class of u.e.i. ceers is properly contained in the class of e.i. ceers (by showing that there is an e.i. ceer that is not universal, whereas all u.e.i. ceers are universal). This conclusion is also a consequence of the previous theorem:

**Corollary 3.11.** *The u.e.i. ceers form a proper subclass of the e.i. ceers.*

*Proof.* The claim follows immediately by the fact that the index set of the u.e.i. ceers is  $\Sigma_3^0$ , whereas the index set of the e.i. ceers is  $\Pi_4^0$  complete.  $\square$

#### 4. THE COMPLEXITY OF $\leq_c$ ITSELF

An obvious generalization of computable reducibility from equivalence relations to pre-orders is the following: Given pre-orders  $R, S$  on the natural numbers, we say that  $R$  is *computably reducible* (or, simply, *reducible*) to  $S$  (notation:  $R \leq_c S$ ) if there is a computable function  $f$  such that, for all  $x, y$ ,  $x R y$  if and only if  $f(x) S f(y)$ . Recently Ianovski, Miller, Nies and Ng [12] have used this reducibility to classify the complexity of several pre-orders which appear in mathematics and

computability theory. For instance they show that the pre-order  $\leq$ , where  $i \leq j$  if  $W_i \leq_T W_j$ , is  $\Sigma_3^0$  complete.

In this section we prove that the reducibility  $\leq_c$  on ceers induces a  $\Sigma_3^0$  complete pre-order on numbers, where we write  $i \leq_c j$  if  $R_i \leq_c R_j$ . This will follow from the next result, which in turn shows that the pre-ordering relation  $\leq_1$  on numbers induced by 1-reducibility on c.e. sets (for which we write  $i \leq_1 j$  if  $W_i \leq_1 W_j$ ) is  $\Sigma_3^0$  complete.

**Theorem 4.1.**  $\leq_1$  is a  $\Sigma_3^0$  complete pre-order: in fact, for any given  $\Sigma_3^0$  pre-order  $\leq$ , there is a computable function  $f$  so that  $W_{f(i)}$  is infinite for all  $i$  and

$$(\forall i, j)[i \leq j \Leftrightarrow W_{f(i)} \leq_1 W_{f(j)}].$$

*Proof.* It is straightforward to check that  $\leq_1$  is  $\Sigma_3^0$ . Let  $\leq$  be a  $\Sigma_3^0$  complete pre-order. We construct a uniform enumeration of  $W_{f(a)}$  for each  $a$  as follows. Since  $\leq$  is  $\Sigma_3^0$ , as in the proof of Theorem 2.1, we can fix a recursive  $g$  so that

$$a \leq b \Leftrightarrow (\exists k)[W_{g(a,b,k)} = \omega].$$

**Requirements and their strategies.** We have requirements:

$$\begin{aligned} Q_{ij}^k &: W_{g(i,j,k)} = \omega \Rightarrow W_{f(i)} \leq_1 W_{f(j)}; \\ P_{ij}^k &: (\forall l \leq k)[W_{g(i,j,l)} \neq \omega] \Rightarrow [\varphi_k \text{ does not } m\text{-reduce } W_{f(i)} \text{ to } W_{f(j)}]; \\ I_i^k &: \text{the set } W_{f(i)} \text{ contains at least } k \text{ elements.} \end{aligned}$$

Let us fix a priority ordering on the requirements. We now outline the strategies to meet the requirements.

*Q-requirements.* A  $Q_{ij}^k$ -requirement builds a computable set  $A_{i,j}^k$  as follows: whenever  $\min(\omega \setminus W_{g(i,j,k)})$  increases, it adds a new element  $a$  to  $A_{i,j}^k$ . At such stages, if this is the  $m^{\text{th}}$  element (i.e.,  $a = a_{i,j}^k(m)$ , where we write  $a_{i,j}^k(n)$  for the  $n^{\text{th}}$  element of  $A_{i,j}^k$ ), and some  $n < m$  is enumerated into  $W_{f(i)}$ , then the strategy enumerates  $a_{i,j}^k(n)$  into  $W_{f(j)}$ . As such, if there are infinitely many stages where  $\min(\omega \setminus W_{g(i,j,k)})$  increases (and no higher priority requirement ruins the coding), then  $n \mapsto a_{i,j}^k(n)$  is a 1-reduction of  $W_{f(i)}$  into  $W_{f(j)}$ .

*P-requirements.* A  $P_{ij}^k$ -requirement acts as follows: to diagonalize and ensure that  $\varphi_k$  is not an  $m$ -reduction, we pick  $x$  larger than any element mentioned before. We wait for  $\varphi_k(x)$  to converge. If it converges to an element which lies already in  $W_{f(j)}$ , then we restrain  $x$  out of  $W_{f(i)}$ . If it converges to an element not restrained out of  $W_{f(j)}$  by any higher priority requirement, we enumerate  $\varphi_k(x)$  into  $W_{f(j)}$  and do not enumerate  $x$  into  $W_{f(i)}$  (again, we place a restraint against this). We now suppose that  $\varphi_k(x)$  is restrained out of  $W_{f(j)}$  for a higher-priority requirement: suppose it is restrained due to being a witness chosen for a higher priority  $P$ -requirement. Then  $P_{ij}^k$  simply enumerates  $x$  into  $W_{f(i)}$ . If, later,  $\varphi_k(x)$  is enumerated into  $W_{f(j)}$ , then that higher-priority  $P$ -requirement will have injured  $P_{ij}^k$ , which we allow. Now suppose  $\varphi_k(x)$  is restrained due to being in the set  $A_{i',j}^{k'}$  for a higher-priority  $Q_{i',j}^{k'}$ -requirement. Suppose it is the  $n^{\text{th}}$  element of the set  $A_{i',j}^{k'}$ , i.e.,  $\varphi_k(x) = a_{i',j}^{k'}(n)$ , and  $n \neq x$  or  $i' \neq i$ . We then put  $\varphi_k(x)$  into  $W_{f(j)}$  and  $n$  into  $W_{f(i')}$ , and we restrain  $x$  out of

$W_{f(i)}$ . In a subsequent paragraph we will analyze in more detail how  $P_{ij}^k$  interacts with several higher priority requirements, and how to deal with the case  $n = x$  and  $i' = i$ .

*I-requirements.* An  $I_i^k$  requirement simply selects new unrestrained elements and enumerates them into  $W_{f(i)}$  to ensure  $W_{f(i)}$  has size at least  $k$ .

*The environments.* At stage  $s$  of the construction, we use several parameters. A  $Q$ -requirement  $Q_{i,j}^k$  uses the parameters  $A_{i,j}^k(s)$ ,  $a_{i,j}^k(n, s)$ , approximating respectively the set  $A_{i,j}^k$  and the witness, coding whether or not  $n$  is in  $W_{f(i)}$ , as in the informal description of the strategy for  $Q_{i,j}^k$ ; in other words, the mapping  $n \mapsto a_{i,j}^k(n, s)$  approximates a computable function that 1-reduces  $W_{f(i)}$  to  $W_{f(j)}$ . After the last initialization of  $Q_{i,j}^k$  (if eventually it stops being re-initialized), whenever we define  $a_{i,j}^k(m, s)$ , for some  $m$ , then this will be also the last value  $a_{i,j}^k(m) = a_{i,j}^k(m, s)$ . Notice that without loss of generality we may assume

$$n < a_{i,j}^k(n, s).$$

A  $P$ -requirement  $P_{i,j}^k$  uses the parameter  $x_{i,j}^k(s)$ , which approximates the witness  $x$ , as described in the above description of the strategy for  $P_{i,j}^k$ . For each  $i, j, k$ ,  $P_{i,j}^k$  also uses a parameter  $S_{i,j}^k(s)$ , which is a finite set of numbers representing the restraint that these numbers not enter  $W_{f(i)}$ . For every  $i$ , we also define a finite approximation  $V_{i,s}$  to  $W_{f(i)}$ , such that  $\{V_{i,s} \mid s \in \omega\}$  is a computable approximation to  $W_{f(i)}$ .

*Interaction of  $P_{i,j}^k$  with more than one requirement.* We now need to analyze in detail what happens when we want to act for  $P_{i,j}^k$  at a stage when  $\varphi_k(x)$ , with  $x = x_{i,j}^k$ , has not as yet been enumerated into  $W_{f(j)}$ , and in fact is restrained out of  $W_{f(j)}$  for a higher-priority requirement  $\mathcal{R}$ . Assume that  $\varphi_k(x)$  converges to, say,  $y$ . If  $\mathcal{R} = P_{j,i_0}^{k_0}$ , for some  $i_0, k_0$ , and we have that  $y = x_{j,i_0}^{k_0}$ , then, as already observed, the conflict is just solved by priority: we enumerate  $x$  in  $W_{f(i)}$ , and if  $\mathcal{R}$  acts, then  $\mathcal{R}$  initializes  $P_{i,j}^k$ .

The problematic case is when there are  $j_1, k'_0$ , and  $y_1$ , such that  $\mathcal{R} = Q_{j_1,j}^{k'_0}$ , and  $y = a_{j_1,j}^{k'_0}(y_1)$ : then we are able to act as desired, i.e. enumerate  $y$  into  $W_{f(j)}$ , but at the same time keeping correctness of  $a_{j_1,h}^{k'_0}(y_1)$ , only if there is no restraint in enumerating also  $y_1$  into  $W_{f(j_1)}$ .

Now in turn, a restraint on  $y_1$  can have been put either by a higher priority  $P_{j_1,i_1}^{k_1}$ , if  $y_1 = x_{j_1,i_1}^{k_1}$ , but then again the conflict is solved, as above, by priority; or,  $y_1$  is restrained by a higher priority  $Q_{j_2,j_1}^{k'_1}$ , if  $y_1$  is of the form  $y_1 = a_{j_2,j_1}^{k'_1}(y_2)$ .

This suggests the following definition:

**Definition 4.2.** Define the sequence  $y_0, y_1, \dots, y_h, \dots$  by steps:

*Step 0:* Let  $y_0 = y$ , and  $j_0 = j$ .

*Step 1:* If there is no restraint on  $y_0$ , or there are unique  $i_0, k_0$  such that  $y_0 = x_{j_0,i_0}^{k_0}$ , then  $y_1$  is undefined; otherwise there exist unique  $j_1, k'_0, y_1$  such that  $y_0 = a_{j_1,j_0}^{k'_0}(y_1)$ ;

*Step  $h + 1$ :* If there is no restraint on  $y_h$ , or there are unique  $i_h, k_h$  such that  $y_h = x_{j_h,i_h}^{k_h}$  then  $y_{h+1}$  is undefined; otherwise there exist unique  $j_{h+1}, k'_h, y_{h+1}$  such that  $y_h = a_{j_{h+1},j_h}^{k'_h}(y_{h+1})$ .



Notice that at each step of the above inductive definition, the various disjuncts are exclusive: this claim (and the claims on uniqueness of  $j_h, i_h, k_h, k'_h$ ) are justified (see Lemma 4.4) by the fact that strategies for different requirements use disjoint sets of witnesses and numbers.

**Lemma 4.3.** *The sequence  $y_0, y_1, \dots, y_h, \dots$  is finite.*

*Proof.* For every  $r$ , if

$$y_r = a_{j_{r+1}, j_h}^{k'_r}(y_{r+1})$$

then  $y_{r+1} < y_r$ . Thus the sequence must terminate.  $\square$

As currently  $y \notin V_j$ , and assuming correctness of the various functions  $a_{j_r, i_r}^{k'_r}(\cdot)$  relative to higher priority requirements, we have that, for every  $r$ ,  $y_r \notin V_{j_r}$ . So the strategy for  $P_{i,j}^k$  in relation to restraints posed by higher priority requirements is the following:

- (1) if the last entry of the sequence is  $y_h$  with  $y_h \in S_{j', i'}^{k'}$  where  $P_{j', i'}^{k'}$  has higher priority, then enumerate  $x_{i,j}^k$  into  $V_i$ ; we have  $x_{i,j}^k \in V_i$ , but  $y = \varphi_k(x_{i,j}^k) \notin V_j$ , unless  $P_{j_h, i_h}^{k_h}$  acts and places  $y_h$  into  $V_{j_h}$ , but in this case all requirements of lower priority than  $P_{j', i'}^{k'}$ , including  $P_{i,j}^k$ , are initialized;
- (2) if the last entry of the sequence is  $y_{h+1}$  with  $y_h = a_{j_{h+1}, j_h}^{k'_h}(y_{h+1})$  where  $y_{h+1}$  is not restrained by higher priority requirements and either  $j_{h+1} \neq i$  or  $y_{h+1} \neq x_{i,j}^k$ , then enumerate each  $y_r$  with  $r \leq h+1$  into  $V_{j_r}$ . We have, as desired,  $y = \varphi_k(x_{i,j}^k) \in V_j$ , but  $x_{i,j}^k \notin V_i$ ; our action has not injured the higher priority requirements (in this case, only  $Q$ -requirements) since all relative 1-reductions have been corrected, having (for all  $r \leq h$ )

$$y_{r+1} \in V_{j_{r+1}} \Leftrightarrow a_{j_{r+1}, j_r}^{k'_r}(y_{r+1}) = y_r \in W_{j_r}.$$

In this case, we keep  $x_{i,j}^k$  in  $S_{i,j}^k$  to restrain lower priority requirements from ever causing  $x_{i,j}^k$  to enter  $W_{f(i)}$ .

- (3) if the last entry of the sequence is  $y_{h+1}$  with  $y_h = a_{j_{h+1}, j_h}^{k'_h}(y_{h+1})$  where  $j_{h+1} = i$  and  $y_{h+1} = x_{i,j}^k$ , then we cannot keep  $x_{i,j}^k$  out of  $V_i$  while enumerating  $y$  into  $V_j$ , due to higher priority  $Q$ -requirements. In this case,  $P_{i,j}^k$  adds  $x_{i,j}^k$  to  $S_{i,j}^k$  and then *unassigns*  $x_{i,j}^k$  (and will thus choose a new  $x_{i,j}^k$  when acting next). We will argue below, using the fact that  $\leq$  is a pre-order, that if  $P_{i,j}^k$  is injured infinitely often in this way, then  $i \leq j$ .

**Construction.** At stage  $s+1$  we may enumerate new elements into some of the sets  $\{V_{i,s} : i \in \omega\}$ , thus obtaining their new approximations  $\{V_{i,s+1} : i \in \omega\}$ . We may also update the definition of some of the parameters. It is understood that if  $V_i$ , or a parameter, is not updated then its value is the same as at the previous stage.

At the end of a given stage  $s$ , we may *initialize* a requirement  $\mathcal{R}$ : For this, if  $\mathcal{R} = Q_{i,j}^k$ , then we set  $A_{i,j}^k(s) = \emptyset$ , and each  $a_{i,j}^k(n, s)$  to be undefined; if  $\mathcal{R} = P_{i,j}^k$ , then we set  $x_{i,j}^k(s)$  to be undefined and  $S_{i,j}^k(s) = \emptyset$ .

We say that a requirement  $\mathcal{R}$  *requires attention* at stage  $s > 0$ , if  $\mathcal{R}$  has not acted since last being initialized, or

- (1)  $\mathcal{R} = Q_{i,j}^k$  and  $s$  is  $\langle i, j, k \rangle$ -*expansionary*, i.e.,  $\min(\omega \setminus W_{g(i,j,k),s}) > \min(\omega \setminus W_{g(i,j,k),\ell})$  where  $\ell$  is the last stage where  $Q_{i,j}^k$  acted; or
- (2)  $\mathcal{R} = P_{i,j}^k$  and either  $x_{i,j}^k(s)$  is not defined or  $\varphi_{k,s}(x_{i,j}^k(s))$  converges and

$$x_{i,j}^k(s) \in V_{i,s} \Leftrightarrow \varphi_{k,s}(x_{i,j}^k(s)) \in V_{j,s}.$$

At odd stages, we take care of  $P$ -requirements and  $Q$ -requirements. At nonzero even stages, we take care of the  $I$ -requirements.

*Stage 0.* Initialize all requirements.

*Stage  $2s+1$ .* Let  $\mathcal{R}$  be the least  $P$ - or  $Q$ -requirement that requires attention. (Notice that cofinitely many such requirements have never acted.) We say that  $\mathcal{R}$  *acts* at  $2s+1$ . For simplicity in the following, when writing down the various parameters, we do not explicitly mention the stage  $s$ .

- (1) If  $\mathcal{R} = Q_{i,j}^k$ , then pick a new element  $a$  and place it into  $A_{i,j}^k$ : if  $a$  is the  $m$ -th element of  $A_{i,j}^k$  in order of magnitude then define  $a = a_{i,j}^k(m)$ . For all  $n < m$ , if  $n \in V_i$ , then enumerate  $a_{i,j}^k(n)$  into  $V_j$ .
- (2) If  $\mathcal{R} = P_{i,j}^k$  then
- (a) if  $x_{i,j}^k$  is not defined, then define it to be a new element and add  $x_{i,j}^k$  to  $S_{i,j}^k$ ;
  - (b) if  $\varphi_k(x_{i,j}^k)$  converges and  $\varphi_k(x_{i,j}^k) \notin V_j$ , then consider the sequence  $y_0, y_1, \dots, y_h, \dots$  of Definition 4.2 (approximated at stage  $2s+1$ ):
    - (i) if the last entry of the sequence is  $y_h = x_{i',j'}^{k'}$  with  $y_h \in S_{i',j'}^{k'}$  where  $P_{i',j'}^{k'}$  has higher priority, then enumerate  $x_{i,j}^k$  into  $V_i$ , remove  $x_{i,j}^k$  from  $S_{i,j}^k$ , and initialize all lower priority requirements;
    - (ii) if the last entry of the sequence is  $y_{h+1}$  with  $y_h = a_{j_{h+1},j_h}^{k'_h}(y_{h+1})$  where  $j_{h+1} \neq i$  or  $y_{h+1} \neq x_{i,j}^k$ , then enumerate each  $y_r$  with  $r \leq h+1$  into  $V_{j_r}$  and initialize all lower priority requirements;
    - (iii) if the last entry of the sequence is  $y_{h+1}$  with  $y_h = a_{j_{h+1},j_h}^{k'_h}(y_{h+1})$  where  $j_{h+1} = i$  and  $y_{h+1} = x_{i,j}^k$ , then unassign  $x_{i,j}^k$ .

Go to Stage  $2s+2$ .

**Stage  $2s+2$ .** If  $s = \langle i, k \rangle$ , and  $V_i$  has less than  $k$  elements, then choose new numbers and enumerate them into  $V_i$ , so that the set has at least  $k$  elements.

This ends the construction.

**Verification.** It is left to verify that the construction works.

The following Lemma observes that in case (2bi), there is never any injury to enumerating  $x_{i,j}^k$  into  $V_i$  and in case (2bii), there is never any injury to enumerating the  $y_r$  into  $V_{j_r}$ .

**Lemma 4.4.** *For any  $i, j, j', k, k'$ , if  $(j, k) \neq (j', k')$ , then  $x_{i,j}^k$  is never in  $S_{i,j'}^{k'}$ . There is never an element  $a_{i,j}^k(y)$  in  $S_{j,j'}^{k'}$  for any  $i, j, j', k, k'$ .*

*Proof.* Each time  $x_{i,j}^k$  is chosen, it is chosen to be a new number, and a number enters  $S_{i,j}^{k'}$  only after it has already been  $x_{i,j}^{k'}$ . Each time  $a_{i,j}^k(y)$  is chosen and each time  $x_{j,j'}^{k'}$  is chosen, they are chosen to be new numbers, and no number enters  $S_{j,j'}^{k'}$  unless it has already been  $x_{j,j'}^{k'}$ .  $\square$

**Lemma 4.5.** *No  $P$ -requirement initializes lower-priority requirements infinitely often.*

*Proof.* Let  $\mathcal{R}$  be a  $P$ -requirement. Suppose, by induction, none of the higher-priority  $P$ -requirement initializes lower-priority requirements infinitely often. Let  $s$  then be a stage after which  $\mathcal{R}$  is never initialized by a higher-priority requirement. If, after stage  $s$ ,  $\mathcal{R}$  ever initializes lower-priority requirements, it is through case (2bi) or (2bii). In either case, then  $\mathcal{R}$  never acts again, so it can initialize lower-priority requirements at most once after stage  $s$ .  $\square$

**Lemma 4.6.** *If  $i \not\leq j$  then  $P_{i,j}^k$  is satisfied.*

*Proof.* Let  $s$  be a stage when  $P_{i,j}^k$  is never initialized by a higher-priority requirement after stage  $s$ . We first argue that  $\mathcal{R}$  cannot be initialized via (2biii) infinitely many times. Suppose otherwise. Then, each time it is initialized, consider the sequence  $j_0, j_1, \dots, j_{h+1}$  where  $j_{h+1} = i$ . Let  $a_0, a_1, \dots, a_n$  be a *simple* sub-path (i.e., if  $j_m$  and  $j_n$  are equal, then we replace the sequence  $j_0, \dots, j_m, \dots, j_n, \dots, j_{h+1}$  by the sequence  $j_0, \dots, j_m, j_{n+1}, \dots, j_{h+1}$ , and repeat this algorithm until all the elements of the sequence are distinct). By the pigeonhole principle, for infinitely many initializations, this sequence  $a_0, \dots, a_n$  is the same. But then, the requirements  $Q_{a_{m+1}, a_m}^{k_m}$  are acting infinitely often. Thus, using that  $\leq$  is a preorder,  $a_{m+1} \leq a_m$  for each  $m \leq n$ , and thus  $i \leq j$ .

Thus, we can consider a stage  $t > s$  such that  $P_{i,j}^k$  is never initialized after stage  $t$ . Let  $x = x_{i,j}^k$  at some stage after  $t$ . This is the final value of  $x_{i,j}^k$ . Subsequently, either  $\varphi_k(x_{i,j}^k)$  diverges, in which case  $P_{i,j}^k$  does not act anymore, and is satisfied as  $\varphi_k$  is not total; or,  $\varphi_k(x_{i,j}^k)$  converges. In this latter case, it either never acts, in which case  $\varphi_k(x_{i,j}^k) \in W_{f(j)}$ , but since  $x_{i,j}^k \in S_{i,j}^k$ , we have that  $x_{i,j}^k \notin W_{f(i)}$ , so  $P_{i,j}^k$  is satisfied; or it acts once more through (2bi), in which case  $x_{i,j}^k \in W_{f(i)}$ , but  $\varphi_k(x_{i,j}^k) \notin W_{f(j)}$ ; or it acts through (2bii): in this case we get  $x_{i,j}^k \notin W_{f(i)}$ , and  $\varphi_k(x_{i,j}^k) \in W_{f(j)}$ . In all cases,  $P_{i,j}^k$  is satisfied.  $\square$

**Lemma 4.7.** *If  $i \not\leq j$ , then  $W_{f(i)}$  does not  $m$ -reduce to  $W_{f(j)}$ .*

*Proof.* By Lemma 4.6, every  $P_{i,j}^k$  is satisfied.  $\square$

**Lemma 4.8.** *If  $i \leq j$ , then  $W_{f(i)} \leq_1 W_{f(j)}$ .*

*Proof.* Let  $k$  be least number such that  $W_{g(i,j,k)} = \omega$ . By Lemma 4.5, every  $P$ -requirement of priority higher than  $Q_{i,j}^k$  initializes  $Q_{i,j}^k$  only finitely often. After the last time  $Q_{i,j}^k$  is initialized, every time  $Q_{i,j}^k$  acts, it defines more and more values of the coding function  $a_{i,j}^k(\cdot)$ , and keeps it correct as a 1-reducibility, by putting  $a_{i,j}^k(n)$  into  $W_{f(j)}$  if and only if  $n \in W_{f(i)}$ .  $\square$

**Lemma 4.9.** *For every pair  $i, k$ , the requirement  $I_i^k$  is satisfied.*

*Proof.* The proof is trivial.  $\square$

$\square$

$\square$

We are now ready to show that the pre-order  $\leq_c$  on indices of ceers is  $\Sigma_3^0$  complete.

**Corollary 4.10.**  $\leq_c$  is a  $\Sigma_3^0$  complete pre-order.

*Proof.* It is straightforward to check that  $\leq_c$  is  $\Sigma_3^0$ . Since for infinite c.e. sets  $X, Y$ ,  $R_X \leq_c R_Y$  if and only if  $X \leq_1 Y$  (where  $R_X$  is the ceer where  $aR_X b$  if and only if  $a = b$  or  $a, b \in X$ . See e.g. [1, 4, 18, 6, 9]) the above reduction allows us to reduce  $\leq$  into  $\leq_c$  as well.  $\square$

The following corollaries are immediate consequence of Theorem 4.1, the first of which appears in [7]:

**Corollary 4.11** ([7]). *The equivalence relation  $\equiv_1$  is a  $\Sigma_3^0$  complete equivalence relation.*

*Proof.* Trivial by Theorem 4.1, since an equivalence relation is a symmetric pre-ordering relation.  $\square$

**Corollary 4.12.**  $\equiv_c$  is a  $\Sigma_3^0$  complete equivalence relation.

*Proof.* Trivial by Corollary 4.10.  $\square$

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