

New Spectra of Strongly Minimal Theories in Finite Languages

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Abstract

We describe strongly minimal theories T_n with finite languages such that in the chain of countable models of T_n , only the first n models have recursive presentations. Also, we describe a strongly minimal theory with a finite language such that every non-saturated model has a recursive presentation.

1. Introduction

Given an \aleph_1 -categorical non- \aleph_0 -categorical theory T in a countable language, the Baldwin-Lachlan theorem [2] says that the countable models of T form an $\omega + 1$ -chain: $\mathcal{M}_0 \prec \mathcal{M}_1 \prec \dots \prec \mathcal{M}_\omega$. We define the *spectrum of recursive models of T* to be $SRM(T) = \{i \mid \mathcal{M}_i \text{ has a recursive presentation}\}$. The spectrum problem asks “Which subsets of $\omega + 1$ can occur as spectra of \aleph_1 -categorical theories?”, and of particular interest is which subsets of $\omega + 1$ can occur as spectra of strongly minimal theories.

There have been various contributions to the spectrum problem over the years. Many have been of the form “ S is a possible spectrum achieved with a strongly minimal (or simply \aleph_1 -categorical) theory”. In this paper, the goal is to achieve many of the same spectra while using a theory in a finite language. This goal has its roots in Herwig, Lempp, Ziegler [3], where it is shown that $\{0\}$ is a possible spectrum using only a finite language. In [1], we show that $\{\omega\}$ is a possible spectrum using only a finite language. Here we show that $\{0, \dots, n\}$ for $n \in \omega$ are possible spectra with a finite language and also that ω is a possible spectrum in a finite language.

In section 3, we demonstrate a new variant of the Hrushovski amalgamation method. The new content of section 3 is confined to the use of an $f_B(A)$ term in the bounding function μ . This allows us to code information into the type of a tuple depending upon how independent the tuple is. As the Hrushovski method is very closely followed to ensure a resulting strongly minimal theory, many of the lemmas and the general outline of the proof are as in [4]. Nonetheless, they are repeated here along with many proofs so that this paper may be read without having read [4].

2. Background

That $\{0, \dots, n\}$ and ω are possible spectra are results from [6] and [5]. In this section, we will review the coding apparent in those constructions. In what follows, we use K to refer to a complete Σ_1 set and use K_s to refer to the part of K enumerated by stage s .

A theory where $SRM(T) = \{0, \dots, n\}$:

Let $L = \{R_i \mid i \in \omega\} \cup \{c_j \mid j \in \omega\}$ where R_i are all $n + 1$ -ary relation symbols. Let M be a model with universe ω where the element $l \in \omega$ is named by the constant c_l , and $R_i(\bar{x}) \leftrightarrow$ the x_j are distinct and $i \in K_s$ where $s = \min\{\bar{x}\}$. The model of dimension k contains k elements that are not named by constants. In any model M with $\bar{y} \subseteq M$ being $n + 1$ elements not named by constants, we see that $M \models R_i(\bar{y}) \Leftrightarrow i \in K$. Since K is non-recursive, this model cannot have a recursive presentation. Thus $m > n \rightarrow m \notin SRM(T)$.

If in a model N every distinct $n + 1$ -tuple contains an element named by a constant, then we can recursively answer which of the R_i should hold on which tuples, and thus the model is recursive. Thus $m \leq n \rightarrow m \in SRM(T)$.

A theory where $SRM(T) = \omega$:

Let $L = \{R_{k,s} | k, s \in \omega\} \cup \{c_j | j \in \omega\}$ where $R_{k,s}$ is a k -ary relation. Fix a complete Π_2 set $S = \{k | \forall l \exists j \phi(k, l, j)\}$. Let M be a model with universe ω where the element $l \in \omega$ is named by the constant c_l and $R_{k,s}(\bar{x})$ holds if and only if the x_i are distinct and $\forall n \leq s \exists j \leq B\phi(k, n, j)$ where $B = \min\{\bar{x}\}$.

Again we see that a model of dimension k has k elements not named by constants. We see similarly that any non-infinite dimensional model is recursive. Let M_ω be the model of dimension ω . Then we see that

$$k \in S \text{ if and only if } \exists \bar{y} (\forall s M_\omega \models R_{k,s}(\bar{y}))$$

But then if M_ω were recursive, S would be a Σ_2 set, which is it not.

The goal of this paper is to use an alteration of the Hrushovski construction to construct a strongly minimal theory where we code sets into models in ways very similar to these while using a finite language.

3. Altering the Hrushovski Construction

We fix L to be the language generated by a single ternary relation symbol. In generality, the following construction will work for any language with a finite relational signature. In the amalgamation that follows, the most natural form of amalgamation will be the free-join.

Definition 1. Let $\{B_i\}_{i \in I}$ be a collection of finite L -structures whose pairwise intersection is A . We say $\bigcup_{i \in I} B_i$ is a free-join over A if whenever $R(\bar{a})$ and $\bar{a} \subseteq \bigcup_{i \in I} B_i$, then $\bar{a} \subseteq B_i$ for some i .

The core idea in Hrushovski's amalgamation construction for building strongly minimal sets is to use a pre-dimension function to give a coherent notion of what algebraicity should be in the constructed theory. We fix the pre-dimension function $\delta(A) = |A| - |R(A)|$ where $|R(A)|$ is the number of triples from A on which R holds. In general, a pre-dimension function can be any function δ from finite L -structures to $\mathbb{Z} \cup \{-\infty\}$ with the following properties.

1. For any finite L -structures A and B , $\delta(A \cup B) \leq \delta(A) + \delta(B) - \delta(A \cap B)$
2. For M any finite L -structure and $B_1, B_2 \subseteq M$, $\delta(B_1 \cup B_2) = \delta(B_1) + \delta(B_2) - \delta(B_1 \cap B_2)$ if and only if $B_1 \cup B_2$ is the free-join of B_1 and B_2 over $B_1 \cap B_2$ in M .

Note that our δ satisfies these properties.

The following definition and combinatorial lemmas are standard for Hrushovski amalgamation constructions and can be found in [4].

Definition 2. For any finite L -structures A and B and infinite L -structure D , we define:

- $\delta(B/A) = \delta(A \cup B) - \delta(A)$. This is the relative dimension of B over A .
- If $A \subseteq B$, we set $\delta(A, B) = \min\{\delta(C) | A \subseteq C \subseteq B\}$. This is the dimension of A in B .
- If $A \subseteq B$, we say A is strong in B or $A \leq B$ if $\delta(A) = \delta(A, B)$.
We say A is strong in D if $A \subseteq D$ and A is strong in C for each finite $A \subseteq C \subseteq D$.

- We say B is simply algebraic over A if $B \neq \emptyset$, $A \cap B = \emptyset$, $A \leq A \cup B$, $\delta(B/A) = 0$, and there is no proper subset B' of B such that $\delta(B'/A) = 0$.
- We say that B is minimally simply algebraic over A if B is simply algebraic over A and there is no proper subset A' of A such that B is simply algebraic over A' .

We verify that strongness forms a transitive reflexive relation, justifying the use of the symbol \leq . Also, we verify that relative dimension acts as we expect.

Lemma 3. *Let $A \subseteq N$ be L -structures. Suppose $A \leq N$*

1. $\delta(X \cap A) \leq \delta(X)$ whenever $X \subseteq N$.
2. $\delta(A', A) = \delta(A', N)$ whenever $A' \subseteq A$.
3. In particular, if $A' \leq A \leq N$, then $A' \leq N$

Proof. 3 is immediate from 2, which in turn is immediate from 1, so we will only prove 1. $\delta(X \cup A) \leq \delta(X) + \delta(A) - \delta(X \cap A)$. So, $0 \leq \delta(X \cup A) - \delta(A) \leq \delta(X) - \delta(X \cap A)$. \square

Lemma 4. *If X, A , and B are finite L -structures such that $A \subseteq B$, then $\delta(X/A \cup (X \cap B)) \geq \delta(X/B)$. In particular, if $X \cap B = \emptyset$, then $\delta(X/A) \geq \delta(X/B)$.*

Proof. $\delta((X \cup A) \cup B) \leq \delta(X \cup A) + \delta(B) - \delta((X \cup A) \cap B)$, which simplifies to $\delta(X \cup B) - \delta(B) \leq \delta(X \cup A) - \delta(A \cup (X \cap B))$, as needed. \square

Lemma 5. *Let M be a finite L -structure. Let $A \subseteq M$ and suppose B_j are simply algebraic over A and $A \leq (A \cup \bigcup_j B_j)$, ($j \in J$). Then:*

1. The B_j are pairwise equal or disjoint.
2. $A \cup \bigcup_j B_j$ is a free join of the B_j over A .
3. Suppose $A \subseteq A' \subseteq M$, $A' \leq A' \cup B_j$, and B_j is not a subset of A' ($j=1,2$). Then any isomorphism of B_1 with B_2 over A extends to an isomorphism over A' . In fact, $A' \cup B_j$ is a free join of A' and B_j over A .

Proof. The proof follows via repeated use of the properties of pre-dimension functions, and can be found as Lemma 2 in [4] \square

Our amalgamation class will have two restrictions, one forcing hereditary positive dimension and a second bounding the numbers of minimally simply algebraic extensions. The first is seen in the definition of \mathcal{C}_0 .

Definition 6. *Let \mathcal{C}_0 be the class of finite L -structures C such that if $A \subseteq C \in \mathcal{C}_0$, then $\delta(A) \geq 0$.*

Lemma 7. *Suppose $A, B_1, B_2 \in \mathcal{C}_0$, $A = B_1 \cap B_2$, and $A \leq B_1$. Let E be the free-join of B_1 with B_2 over A . Suppose C^1, \dots, C^r, F are disjoint substructures of E such that each C^i is minimally simply algebraic over F and the structures C^i and C^j are isomorphic over F for each $1 \leq i, j \leq r$. Then one of the following holds:*

1. One of the C^i is contained in $B_1 \setminus A$ and $F \subseteq A$.
2. Either $F \cup \bigcup_{i=1}^r C^i$ is entirely contained in B_2 or $F \cup \bigcup_{i=1}^r C^i$ is entirely contained in B_1 and one of the C^i is contained in $B_1 \setminus A$.
3. $r \leq \delta(F)$
4. For one C^i , setting $X = (F \cap A) \cup (C^i \cap B_2)$, $\delta(X/X \cap A) < 0$. Further, one of the C^j is contained in $B_1 \setminus A$. (Note that this cannot happen if $A \leq B_2$ by Lemma 3).

Proof. This proof contains most of the combinatorial difficulties of the amalgamation construction and can be seen in the proof of Lemma 3 in [4]. \square

In order to differentiate between various levels of algebraicity for our future coding purposes, unlike the standard Hrushovski construction of a strongly minimal set, we provide the following definition.

Definition 8. For L -structures $A \subseteq B$ with A finite, we define $f_B(A) = \min\{|C| \mid A \subseteq C \subseteq B, \delta(C) < |A|\}$, where we say the min of an empty set is ∞ .

Lemma 9. If $A \subseteq B \subseteq C$ and $B \leq C$, then $f_B(A) = f_C(A)$

Proof. Take $X \subseteq C$ of minimal size with $\delta(X) < |A|$. Then $\delta(X \cap B) \leq \delta(X) < |A|$. Thus $X \subseteq B$ by minimality. \square

One can think of $f_B(A)$ as a measure of how much A looks independent to the set B . We define the bounding μ function similarly to its analog in [4], though we incorporate f into our definition of μ .

Definition 10. Let $\mu(A, B, n)$ be a function from quantifier-free types of finite L -structures A, B and an $n \in \omega \cup \{\infty\}$ to ω so that for all but finitely many $n \in \omega$, $\mu(A, B, n) = \mu(A, B, \infty)$. Furthermore, we demand that $\mu(A, B, n) \geq \delta(A)$ for all triples A, B, n .

Given a pair A, B of finite L -structures, set $h(A, B)$ to be the least $n \geq |A|$ so that $\mu(A, B, m)$ is constant for all $m \geq n$. For $k \in \omega$, we set $g(k) = \max\{h(A, B) \mid |A|, |B| \leq k\}$.

From any such μ function, we define the following amalgamation class:

Definition 11. Let \mathcal{C} be the class of finite L -structures C such that the following hold:

1. $\delta(A) \geq 0$ for all $A \subseteq C$.
2. Suppose $X_i, i = 1, \dots, n, Y$ are disjoint subsets of C so that the X_i are minimally simply algebraic over Y and the X_i are isomorphic over Y . Then $n \leq \mu(Y, X_1, f_C(Y))$.

Unlike in the original construction, μ depends on $f_C(Y)$, which means that it is possible that $B \subset C \in \mathcal{C}$, but $B \notin \mathcal{C}$. The analog here will be that if $B \leq C \in \mathcal{C}$, then $B \in \mathcal{C}$. Despite this difference, we will show that \mathcal{C} leads us to a strongly minimal amalgam.

Definition 12. Let $A \subseteq B$ be L -structures. We say A is n -strong in B if $\delta(A \cup X) \geq \delta(A)$ for all $X \subseteq B$ with $|X| \leq n$.

Lemma 13. If $B \leq C \in \mathcal{C}$, then $B \in \mathcal{C}$. In fact, if B is $g(|B|)$ -strong in C , then $B \in \mathcal{C}$.

Proof. The first condition holds as any subset A of B is a subset of C . Suppose $X_i, i = 1, \dots, n, Y$ are disjoint subsets of B so that the X_i are minimally simply algebraic over Y , and the X_i are isomorphic over Y . Then $n \leq \mu(Y, X_1, f_C(Y))$. Since $f_B(Y) \geq f_C(Y)$, if $f_C(Y) \geq h(Y, X_1)$, then $\mu(Y, X_1, f_C(Y)) = \mu(Y, X_1, f_B(Y))$. So, we may assume there exists a Z of minimal size so that $Y \subseteq Z$ and $\delta(Z) < |Y|$ with $|Z| < h(Y, X_1)$. Since $B \leq B \cup Z$ by assumption, $\delta(B \cap Z) \leq \delta(Z)$ showing that $Z \subseteq B$ by minimality, and so $f_C(Y) = f_B(Y)$. \square

Lemma 14. (Algebraic Amalgamation Lemma) Suppose $A = B_1 \cap B_2$, $A, B_1, B_2 \in \mathcal{C}$, and $B_1 \setminus A$ is simply algebraic over A . Let E be the free-join of B_1 with B_2 over A . Then $E \in \mathcal{C}$ unless one of the following holds:

- $B_1 \setminus A$ is minimally simply algebraic over $F \subseteq A$, and there are $\mu(F, B_1 \setminus A, f_{B_2}(F))$ disjoint copies of $B_1 \setminus A$ in B_2 .
- There is a set $X \subseteq B_2$ such that $X \cap A \not\subseteq X$, and X is isomorphic to a subset of B_1 .
- There are sets $F \subseteq B_1$ and $C \subseteq B_1$ minimally simply algebraic over F so that $\mu(F, C, f_{B_1}(F)) > \mu(F, C, f_E(F))$.

Proof. If $X \subseteq E$, then $\delta(X) = \delta(X \cap B_1) + \delta(X \cap B_2) - \delta(X \cap A) \geq \delta(X \cap B_2) \geq 0$. If there are disjoint $C^i, F \subseteq E$ so that each of the C^i are minimally simply algebraic over F and each (C^i, F) is isomorphic, then by Lemma 7, we need to consider only four cases:

- One of the C^i is $B_1 \setminus A$. As the C^j and F are disjoint, each of the other C^j and F are contained in B_2 . If $r > \mu(F, C^1, f_E(F))$ then there must be $\mu(F, C^1, f_E(F))$ of them contained in B_2 . Since $B_2 \leq E$, $f_E(F) = f_{B_2}(F)$, showing that the first exception in this lemma holds.
- $F \cup \bigcup_{i=1}^r C^i$ is entirely contained in either B_1 or B_2 . Here, $r \leq \mu(F, C^1, f_{B_j}(F))$ as $B_1, B_2 \in \mathcal{C}$. Since $B_2 \leq E$, if $F \cup \bigcup_{i=1}^r C^i \subseteq B_2$, then $r \leq \mu(F, C^1, f_E(F))$ as $f_E(F) = f_{B_2}(F)$. Thus we only need to consider the case where $F \cup \bigcup_{i=1}^r C^i \subseteq B_1$ and $\mu(F, C^1, f_E(F)) < \mu(F, C^1, f_{B_1}(F))$. In this case, the third exception of this lemma holds.
- $r \leq \delta(F)$. In this case $r \leq \delta(F) \leq \mu(F, C^1, f_E(F))$.
- For one C^j , setting $X = (F \cap A) \cup (C^j \cap B_2)$, we see that $\delta(X/X \cap A) < 0$. Further, one of the C^j is contained in $B_1 \setminus A$. This yields the second exception in this lemma.

□

Lemma 15. (*Strong Amalgamation Lemma*) Suppose $A, B_1, B_2 \in \mathcal{C}$ and $A \leq B_i$. Then there exist $D \in \mathcal{C}$ so that $B_2 \leq D$ and $g : B_1 \rightarrow D$ an embedding so that $g(B_1) \leq D$ and $g(A) = id|_A$.

Proof. We may assume there is no B' such that $A \leq B' \leq B_1$. Thus either $B_1 = A \cup \{x\}$ where x is unrelated to A by R or $B_1 \setminus A$ is simply algebraic over A . In the first case, the free-join suffices. In the second case, the free-join fails only if one of the conditions of the last lemma holds. The second and third conditions cannot hold as $A \leq B_2$. Let $F \subseteq A$ be so that $B_1 \setminus A$ is minimally simply algebraic over F . As $A \leq B_1$ and $A \leq B_2$, $f_{B_1}(F) = f_A(F) = f_{B_2}(F)$. If condition 1 holds, then we have $\mu(F, B_1 \setminus A, f_{B_2}(F))$ copies of $B_1 \setminus A$ in B_2 . There must be no more than $\mu(F, B_1 \setminus A, f_{B_2}(F)) - 1$ contained in A , as $B_1 \in \mathcal{C}$ and $f_{B_2}(F) = f_A(F) = f_{B_1}(F)$. As no copy of $B_1 \setminus A$ in B_2 can be partially in A (as $A \leq B_2$), we have one contained in $B_2 \setminus A$ with which to identify $B_1 \setminus A$. This gives us the required amalgamation. □

The above lemma guarantees that there is a generic amalgamation of the class \mathcal{C} , which we call \mathcal{M} . \mathcal{M} is characterized by three properties:

1. \mathcal{M} is countable.
2. For any finite $A \leq \mathcal{M}$, $A \in \mathcal{C}$.
3. Suppose $A \leq \mathcal{M}$, $A \leq B$, and $B \in \mathcal{C}$. Then there is an embedding $g : B \rightarrow \mathcal{M}$ so that $g|_A = id_A$ and $g(B) \leq \mathcal{M}$.

By a standard back-and-forth on strong substructures, and since each finite $A \subset \mathcal{M}$ is a subset of a finite B such that $B \leq \mathcal{M}$, we see that these three properties fully characterize \mathcal{M} up to isomorphism. Showing that \mathcal{M} is strongly minimal will follow a path analogous to the one in [4]. The new content to the proof will be in the change to $3''$ and the necessity of $2'$.

We would like to show that \mathcal{M} is saturated by showing that any elementary extension of \mathcal{M} satisfies properties (1, 2, 3), but properties 2 and 3 are not first order. To handle this problem, we replace 2 and 3 by $2'$, $3'$, and $3''$:

$2'$: For any finite $A \subseteq \mathcal{M}$, if A is $g(|A|)$ -strong in \mathcal{M} , then $A \in \mathcal{C}$.

$3'$: There is an infinite set I with R not holding on any tuple in I such that $A \leq \mathcal{M}$ for all finite $A \subset I$.

$3''$: Suppose $A \subset \mathcal{M}$, $A \leq B$, and $B \setminus A$ is minimally simply algebraic over $F \subseteq A$. Further, suppose that A is $g(|B|)$ -strong in \mathcal{M} . Then there are $\mu(F, B_1 \setminus A, f_{\mathcal{M}}(F))$ distinct realizations of $tp_{q.f.}(B/F)$ over F in \mathcal{M} .

Note that $2'$, $3'$, $3''$ are first order conditions. Note that if A is $g(|B|)$ -strong in \mathcal{M} , then $\mu(F, B_1 \setminus A, f_{\mathcal{M}}(F)) = \mu(F, B_1 \setminus A, f_A(F))$ as in the proof of Lemma 13.

Claim 16. *The conditions (1, 2, 3) are equivalent to the conditions (1, 2', 3', 3'').*

Proof. Assume (1, 2, 3). To see $2'$ from 2, let B be least so that $A \subseteq B \leq \mathcal{M}$. By 2, $B \in \mathcal{C}$, and applying lemma 13 to the pair (A, B) we see that $A \in \mathcal{C}$. $3'$ follows trivially from 3. $3''$ is a consequence of the algebraic amalgamation lemma employed for any A, B , and set C so that $A \subset C \leq \mathcal{M}$. If the free-join of B with C over A is in \mathcal{C} , then 3 implies that we can amalgamate the free-join into \mathcal{M} over C . Otherwise, one of the conditions in the algebraic amalgamation lemma holds. Since A is $g(|B|)$ -strong in \mathcal{M} , the second and third conditions cannot hold, and if the first condition holds, then there are already $\mu(B \setminus A, F, f_{\mathcal{M}}(F))$ many copies of $B \setminus A$ over F in C .

Assume (1, 2', 3', 3''). 2 is formally weaker than $2'$, so it follows immediately. We show 3: Suppose $A \leq \mathcal{M}$, $A \leq B$. We may assume that there is no B' such that $A \leq B' \leq B$. Thus, B is either simply algebraic over A , or $B = A \cup \{x\}$ where x is a singleton unrelated to A . In the latter case, $3'$ gives us an infinite independent sequence from which to choose an embedding of B over A . In the former case, $3''$ guarantees that there is an embedding of B over A exactly as in the strong amalgamation lemma. \square

Corollary 17. *\mathcal{M} is saturated.*

Proof. Let N be any countable model elementarily containing \mathcal{M} . Then since N satisfies (1, 2, 3', 3'') and hence (1, 2, 3), N is isomorphic to \mathcal{M} . Thus there are only countably many types realized in elementary extensions of \mathcal{M} , so there is a countable saturated model elementarily containing \mathcal{M} , which \mathcal{M} must be isomorphic to. \square

Next we will characterize algebraicity in \mathcal{M} and will thereby establish strong minimality of \mathcal{M} . We define $d(A) = \min\{\delta(C) \mid A \subseteq C \subseteq \mathcal{M}, C \text{ finite}\}$. Clearly for any A and x , either $d(xA) = d(A)$ or $d(xA) = d(A) + 1$.

Lemma 18. *If $d(xA) = d(A) + 1$ and $d(yA) = d(A) + 1$, then $(\mathcal{M}, Ax) \cong (\mathcal{M}, Ay)$.*

Proof. Let B be such that $A \subseteq B$ and $\delta(B) = d(A)$. Then $B \leq \mathcal{M}$ and $d(xB) = d(xA) = d(A) + 1$. Thus $xB \leq \mathcal{M}$, and similarly $yB \leq \mathcal{M}$. Using property 3 and a standard back-and-forth along strong substructures, we see that (\mathcal{M}, xB) and (\mathcal{M}, yB) are isomorphic. \square

Lemma 19. *If $d(xA) = d(A)$ then $x \in \text{acl}_{\mathcal{M}}(A)$.*

Proof. Suppose $d(xA) = d(A)$. First, let B be a minimal set such that $A \subseteq B$ and $\delta(B) = d(A)$. This B is referred to in the literature as the intrinsic closure of A . We show that B is algebraic over A in \mathcal{M} . Suppose there were two realizations of the positive quantifier-free type of B over A . Call the second realization B' . Then $\delta(B \cup B') \leq \delta(B) + \delta(B') - \delta(B \cap B') < \delta(B') = d(A)$. The strict inequality is due to B being a minimal set with the properties that $A \subseteq B$ and $\delta(B) = d(A)$. This inequality is a contradiction to the definition of $d(A)$.

Fix E to be a set such that $xA \subseteq E$ and $\delta(E) = d(A)$. Then $\delta(E \cup B) \leq \delta(E) + \delta(B) - \delta(E \cap B)$. If E does not contain B , then $\delta(E \cap B) > d(A)$ by minimality of B . Then $\delta(E \cup B) \leq d(A) + d(A) - \delta(E \cap B) < d(A)$, again a contradiction. Thus E contains B and $d(xB) = d(B)$.

Take a sequence of extensions $B_0, B_1, B_2, \dots, B_n$ such that $B_0 = B$, $B_n = E$, and B_{i+1} is a minimal set such that $B_i \subsetneq B_{i+1} \subseteq E$ and $\delta(B_{i+1}) = d(A)$. Then B_{i+1} is simply algebraic over B_i , say minimally simply algebraic over F_i . Thus B_{i+1} is algebraic over B_i (any two extensions of B_i satisfying the atomic type of $B_{i+1} \setminus B_i$ over B_i must be disjoint and isomorphic to $B_{i+1} \setminus B_i$ over B_i since $B_i \leq \mathcal{M}$, so we explicitly forced there to be no more than $\mu(F_i, B_{i+1} \setminus B_i, f_{B_i}(F_i))$ of these). We conclude that E is algebraic over A . In particular, $x \in \text{acl}_{\mathcal{M}}(A)$. \square

Corollary 20. *\mathcal{M} is strongly minimal.*

Proof. In the previous lemma, we showed that over any set there is a unique non-algebraic 1-type realized in \mathcal{M} . Since \mathcal{M} is saturated, this shows that $Th(\mathcal{M})$ is strongly minimal. \square

4. $SRM(T) = \{0, \dots, m\}$

Fix an integer m . We will construct a theory T so that $SRM(T) = \{0, \dots, m\}$. Recall that we defined K to be the standard complete Σ_1 set, ie: the halting problem. We set K_s to be the part of K enumerated by stage s . We set $K_\infty = K$. Fix a recursive enumeration of all the quantifier free types of minimally simply algebraic extensions over all sets of size $m+1$. We will refer to these as Λ_i , and will say $\Lambda_i(A, B)$ to mean that B is a minimally simply algebraic extension of A of type enumerated as Λ_i . To construct a model, we use the results of the previous section, and we need only define μ .

Definition 21.

$$\mu(A, B, k) = \begin{cases} |A| + 1 & \text{if for all } i, \neg \Lambda_i(A, B) \text{ (ie: } |A| \neq m+1) \\ |A| + 1 & \text{if } \Lambda_i(A, B), \text{ and } i \in K_k \\ |A| + 2 & \text{if } \Lambda_i(A, B), \text{ and } i \notin K_k \end{cases}$$

We employ the previous section, and we thus get a generic model \mathcal{M} , which is saturated and strongly minimal. Let $T = Th(\mathcal{M})$. Now, we verify that $SRM(T)$ is as required.

Claim 22. $k > m \rightarrow k \notin SRM(T)$

Proof. Let N be any model of dimension $> m$. Let \bar{x} be any tuple of size $m+1$ with \bar{x} an independent set in N . Then $i \in K \leftrightarrow \neg \exists^{m+3} \bar{y} (N \models \Lambda_i(\bar{x}, \bar{y}))$. Thus a complete Σ_1 set can be represented as a Π_1 set using an oracle for quantifier-free statements true about N . Thus N cannot be recursive. \square

Claim 23. $k \leq m \rightarrow k \in SRM(T)$.

Proof. If X is a finite L -structure and $\delta(X) \leq m$, then whether $X \in \mathcal{C}$ is a recursive question. This is simply because $f_X(Y)$ is finite for any $m+1$ element set $Y \subseteq X$, so we can compute $\mu(A, B, f_X(A))$ for any $A, B \subseteq X$. To construct the k -dimensional model, we start with M_0 as k elements unrelated by R . At stage s , we take M_{s-1} and we list off the first s possible simply algebraic extensions over subsets of M_{s-1} . Then we check if the associated free-join keeps us in \mathcal{C} . If it does, we pass to the free-join. After doing this for these s possible extensions, we call the result M_s . This yields a model where we have amalgamated every simply algebraic extension possible, in particular we amalgamate B over A for any strong enough A . Thus we get a model of $1, 2', 3''$. By compactness, there is an elementary superstructure satisfying $2', 3', 3''$, and by downward Löwenheim-Skolem, there is an elementary superstructure satisfying $1, 2', 3', 3''$. Thus $\bigcup_s M_s$ is a model of T . Since $\delta(M_i) = k$ for each M_i and $M_0 \leq M_i$ for each M_i , $\bigcup_s M_s$ is the k -dimensional model of T . \square

Thus we have proved the following:

Theorem 24. *There exists a strongly minimal theory in a language with a single ternary relation symbol such that $SRM(T) = \{0, \dots, m\}$.*

5. $SRM(T) = \omega$

We will be employing the same construction as above, so we need only define a new μ function. In order to work with the more complicated recursion theoretic necessities of this proof, we will be using a complete Π_2 set. We fix one now: $S = \{k \mid \forall l \exists j \phi(k, l, j)\}$. Fix a recursive enumeration of all quantifier-free types of minimally simply algebraic extensions (over any set) $\Lambda_{k,s}$ so that each extension $\Lambda_{k,s}$ is over a set of size k . Now we can define the bounding function μ :

Definition 25.

$$\mu(A, B, n) = \begin{cases} |A| + 1 & \text{if } \Lambda_{k,s}(A, B) \text{ and } \forall l \leq s \exists j \leq n \phi(k, l, j) \\ |A| + 2 & \text{if } \Lambda_{k,s}(A, B) \text{ and } \neg \forall l \leq s \exists j \leq n \phi(k, l, j) \end{cases}$$

Note that μ satisfies the required property that all but finitely many integers agree with the value outputted at ∞ .

We employ the construction above, and we thus get a generic model \mathcal{M} , which is saturated and strongly minimal. Let $T = Th(\mathcal{M})$. Now, we need only verify that $SRM(T)$ is as required.

Claim 26. $\omega \notin SRM(T)$.

Proof. Let N be any particular presentation of the saturated model. For any k ,

$$k \in S \leftrightarrow \exists \bar{x} \left((\forall s \neg \exists^{k+2} \bar{y} N \models \Lambda_{k,s}(\bar{x}, \bar{y})) \wedge (\bar{x} \text{ is strong in } N) \right)$$

Then, we see that a complete Π_2 set is Σ_2 (being strong in N is a Π_1 -condition) in a presentation of the quantifier-free diagram of N . Thus N has no recursive presentation. \square

Claim 27. $n \in \omega \rightarrow n \in SRM(T)$.

Proof. Fix $n \in \omega$.

Claim 28. *The set of finite L -structures X such that $\delta(X) \leq n$ and $X \in \mathcal{C}$ is a recursive set.*

Proof. Non-uniformly, fix a finite set of information detailing for each $i \leq n$, whether $i \in S$, and if $i \notin S$ which is the first s so that $\neg \exists j \phi(i, s, j)$.

Given any $A \subseteq X$, either $|A| \leq n$ or $f_X(A)$ is finite. In the latter case, computing μ is recursive, since all the quantifiers are bounded. In the former case, the information we specified tells us how to compute μ when $f_X(A) = \infty$. \square

To construct the n -dimensional model, we start with M_0 as n elements unrelated by R . At stage s , we take M_{s-1} and we list off the first s possible simply algebraic extensions over subsets of M_{s-1} . Then we check if the associated free-join keeps us in \mathcal{C} . If it does, we pass to the free-join. After doing this for these s possible extensions, we call the result M_s . This yields a model where we have amalgamated every simply algebraic extension possible, in particular we amalgamate B over A for any strong enough A . Thus $\bigcup_s M_s$ is a model of $1, 2', 3''$. By compactness and Löwenheim-Skolem, there is an elementary superstructure satisfying $1, 2', 3', 3''$, so $\bigcup_s M_s$ is a model of T . Since $\delta(M_i) = n$ for each M_i and $M_0 \leq M_i$ for each M_i , $\bigcup_s M_s$ is the n -dimensional model of T . \square

Thus we have proved the following:

Theorem 29. *There exists a strongly minimal theory in a language with a single ternary relation symbol such that $SRM(T) = \omega$.*

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