

The Index Set of Uncountably Categorical Theories

Uri Andrews, Tamvana Makuluni

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Abstract

We classify the complexity of the index set of uncountably categorical theories. We show that this index set surprisingly falls at the intermediate stage of being complete for intersections of Π_2 sets with Σ_2 sets.

1 Introduction

One goal of mathematical logic is to determine the complexity of mathematical notions. The methods most often used to measure complexity of a notion are Kleene's arithmetical and analytic hierarchies. A set is Σ_n if it can be described in arithmetic by a formula with n quantifiers beginning with an existential quantifier. A set is Π_n if it is the complement of a Σ_n set. A set is *arithmetical* (classifiable in Kleene's arithmetical hierarchy) if it is Σ_n for some n . The most natural characterizations of uncountable categoricity are non-arithmetical. For example, the Baldwin-Lachlan characterization of uncountable categoricity as ω -stability along with no infinite 2-cardinal formula is non-arithmetical as ω -stability is Π_1^1 -complete. This leads to the question of whether uncountable categoricity is an arithmetical notion and, if so, of which complexity. To completely characterize the complexity of a set, we show that it is complete at some level of the hierarchy (i.e., as complicated as possible for a set at that level). Lempp and Slaman [3] characterized the complexity of \aleph_0 -categoricity and Ehrenfeuchtness, two other natural model theoretic notions, showing that the former is arithmetical and Π_3 -complete, while the latter is non-arithmetical. Most natural mathematical notions which are arithmetical are Σ_n -complete or Π_n -complete for some n . Surprisingly, uncountable categoricity is arithmetical and lies at an intermediate level, being complete for intersections of a Π_2 set and a Σ_2 set (also known as $0'$ -d.r.e. sets).

Recall that a countable theory T is *uncountably categorical* if it has exactly one model up to isomorphism in each uncountable cardinality. To determine the complexity of this notion, we look at the difficulty of determining whether this property holds of a recursive theory T (i.e., a set of first order sentences with an effective procedure to determine membership of any given formula). Because we focus on recursive theories, we restrict our attention to languages that are countable and recursive. Our proof relativizes to show that the set of uncountably categorical theories recursive in A is complete for intersections of Π_2^A and Σ_2^A sets.

We fix the set \mathbb{T} of indices for recursive sets that are complete uncountably categorical theories and determine the complexity of \mathbb{T} in Kleene's arithmetical hierarchy. We show that \mathbb{T} is an intersection of a Π_2 set and a Σ_2 set, and we show its hardness (i.e., that any

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other set S that is also an intersection of a Π_2 set and a Σ_2 set can be reduced to \mathbb{T} by way of a many-one reduction).

To determine that \mathbb{T} is an intersection of a Π_2 set and a Σ_2 set, we give in Section 2 an arithmetical description of \mathbb{T} with this complexity. It is known (see [4]) that the set of indices for infinite recursively enumerable sets is a Π_2 -complete set and that the set of indices for finite recursively enumerable sets is Σ_2 -complete. Thus, to yield the hardness result for \mathbb{T} , we give in Section 3 an effective procedure which on input $(I, F) \in \omega^2$ outputs a theory T such that T is uncountably categorical if and only if W_I is infinite and W_F is finite (where W_n is the n^{th} recursively enumerable set).

Formulae are allowed to have parameters, unless otherwise stated. A definable set is strongly minimal if every definable subset is either finite or cofinite in it. A strongly minimal formula is simply a formula defining a strongly minimal set. Baldwin and Lachlan [1] showed that the strongly minimal formulae play a crucial role in the structure of models of uncountably categorical theories. In particular, a model of an uncountably categorical theory is prime over the realizations of a strongly minimal formula, thus the strongly minimal formula completely ‘controls’ the model. Strongly minimal formulae will similarly play a central role in our analysis.

2 Description

In the solution that follows, we will give a criterion to determine whether a given theory T in a language L is uncountably categorical. In the criterion, we will make reference to a theory T' , in a language L' , which is the elementary diagram of a countable model of T . More specifically, the language L' of T' is the language generated by L along with countably many new constant symbols, and T' is the elementary diagram of the model of T produced by the Henkin construction. As the Henkin construction is effective, this gives a universal procedure for turning a theory T into an elementary diagram of a countable model of T . In the event that the initial theory is incomplete, T' will simply also be incomplete. On the other hand, if T is complete then so is T' , and T is uncountably categorical if and only if T' is uncountably categorical. This can be seen, for example, as a consequence of Erimbetov’s theorem below or alternatively from the Baldwin-Lachlan characterization of uncountable categoricity as ω -stability along with the non-existence of an infinite 2-cardinal formula.

The benefit of working with T' is simply that if T is uncountably categorical then there is a strongly minimal formula definable with parameters from the prime model of T [1]. In particular, if T is uncountably categorical, there is a \emptyset -definable strongly minimal set in T' .

We use the following theorem describing uncountable categoricity. Recall that a formula ϕ is *1-cardinal* in T if the set of realizations of ϕ in a model M of T always has the same cardinality as M . A formula is *2-cardinal* if it is not 1-cardinal.

Theorem 1 (Erimbetov [2] 9.5.12). *A complete theory T is uncountably categorical if and only if T has a 1-cardinal strongly minimal formula with parameters from any model of T .*

Corollary 2. *A complete theory T is uncountably categorical if and only if T' has a 1-cardinal strongly minimal formula without parameters.*

Proof. \rightarrow : Baldwin and Lachlan [1] showed that the prime model of T has a strongly minimal 1-cardinal formula. In T' this formula is definable without parameters.

\leftarrow : Erimbetov’s theorem shows that this suffices to yield uncountable categoricity of T' , thus uncountable categoricity of T . \square

We now provide the characterization of uncountably categorical theories which demonstrates that \mathbb{T} is a conjunction of a Π_2 set with a Σ_2 set. Using condition 1, condition 2 below simply says that there exists a minimal formula in L' . We write the condition in the less natural way below since this description has lower complexity.

Theorem 3. *A complete first order theory T is uncountably categorical if and only if the following hold:*

1. *T defines no infinite 2-cardinal formula (i.e., each definable set, possibly with parameters, is infinite if and only if it is 1-cardinal).*
2. *There is some $\phi(x) \in L'$ so that $\phi(x)$ is 1-cardinal; and for all $\psi(x) \in L'$ with no parameters, $\psi(x) \wedge \phi(x)$ is 2-cardinal OR $\neg\psi(x) \wedge \phi(x)$ is 2-cardinal.*

Proof. If T is uncountably categorical, then so is T' , and it contains a \emptyset -definable strongly minimal formula, ϕ . Since T' defines no infinite 2-cardinal formulae [1], the first condition holds; and since ϕ is strongly minimal, the second condition holds.

Suppose the two conditions hold. Take ϕ as given by the second condition. Let M be the model of T for which T' is the elementary diagram. Then ϕ is minimal in the model M . As in [1], since there are no infinite 2-cardinal formulae, ϕ being minimal implies that ϕ is strongly minimal. Thus ϕ is a 1-cardinal strongly minimal formula, showing that T' is uncountably categorical. \square

We now verify the complexity of the conditions. Suppose T is recursive, and thus T' is also recursive.

In the following equivalent formulation of condition 1, U is a new unary predicate and \bar{c} are new constant symbols. We write “ $U \prec M$ ” for the first order axiom schema that declares that in each model, U defines a proper elementary substructure of the universe. Condition 1 is equivalent to the following by Vaught [2, Thm 8.4.1]:

$$\forall\phi(x, \bar{y}) [T \cup \text{“}U \prec M\text{”} \cup \{\exists^{\geq n}x \phi(x, \bar{c}) \mid n \in \omega\} \cup \{U(\bar{c})\} \vdash \exists x (\neg U(x) \wedge \phi(x, \bar{c}))]$$

That is, ϕ is 2-cardinal if and only if there exists a pair of models $\bar{c} \in U \prec M$ with $\phi(U, \bar{c}) = \phi(M, \bar{c})$. As the theory

$$T \cup \text{“}U \prec M\text{”} \cup \{\exists^{\geq n}x \phi(x, \bar{c}) \mid n \in \omega\} \cup \{U(\bar{c})\}$$

is uniformly recursive in T and ϕ , the provability of $\exists x (\neg U(x) \wedge \phi(x, \bar{c}))$ from it is a Σ_1 condition. This provability holding for all ϕ is thus Π_2 .

In the following theorem, a *layering* in terms of X is a formula with particular syntactic form so that $\forall x\theta(x)$ demonstrates that X is 1-cardinal. Checking whether θ is a layering in terms of X is recursive. To show that condition 2 is Σ_2 , we use the following theorem:

Theorem 4 (Baldwin-Lachlan, Gaifman [2, 8.4.2]). *A \emptyset -definable set X is 1-cardinal if and only if there exists a layering θ (also a \emptyset -definable formula) in terms of X so that $T \vdash \forall x\theta(x)$.*

Thus, 1-cardinality of a \emptyset -definable set is readily seen to be a Σ_1 condition. Using this, the natural reading of condition 2 shows that it is Σ_2 . Noting that the condition of being a complete theory is also Π_2 , we have shown the following theorem.

Theorem 5. *The index set \mathbb{T} of recursive complete uncountably categorical theories is the intersection of a Π_2 set with a Σ_2 set.*

Corollary 6. *The index set of recursive uncountably categorical theories is the intersection of a Π_2 set with a Σ_2 set.*

Proof. T is uncountably categorical if and only if $T \cup \{\exists^{\geq n} x(x = x) \mid n \in \omega\}$ is complete and uncountably categorical. This provides the required many-one reduction to \mathbb{T} . \square

3 Hardness

We now show that the index set of recursive theories that are uncountably categorical is complete for intersections of Π_2 and Σ_2 sets.

Theorem 7. *The index set of recursive uncountably categorical theories is complete for intersections of Π_2 and Σ_2 sets.*

The index set \mathbb{T} of complete recursive uncountably categorical theories is complete for intersections of Π_2 and Σ_2 sets.

Proof. Since we know these index sets are intersections of Π_2 and Σ_2 sets, we need only show hardness. We use the same reduction for both. On input $(I, F) \in \omega^2$, we produce (uniformly) a complete recursive theory T_{IF} such that T_{IF} is uncountably categorical if and only if W_I is infinite and W_F is finite. Recall that W_n , in addition to being a set of natural numbers, comes with an enumeration of the set W_n in stages that enumerates at most one number per stage. We fix a language with infinitely many unary relation symbols U_j and V_k , as well as infinitely many binary relation symbols R_l .

Each U_j will either do nothing or will split U_{j-1} into two pieces with a bijection provided by R_j . If infinitely many U_j act by splitting U_{j-1} , then there will be continuum many 1-types and the theory will not be uncountably categorical. The V_k will describe disjoint subsets of $\cap_j U_j$. In one possible outcome, the V_k will describe disjoint finite subsets of $\cap_j U_j$, and in the second possible outcome, infinitely many of the V_k will describe infinite disjoint sets. In the second outcome, the infinite V_k will give orthogonal types showing that the theory is not uncountably categorical. If there are only finitely many j such that U_j splits U_{j-1} and all the V_k describe finite sets, then $\cap_j U_j$ will be a strongly minimal 1-cardinal formula, showing that T_{IF} is uncountably categorical.

We now provide a recursive enumeration of axioms for the theory T_{IF} in stages as follows:

Stage 0: We declare that the V_k define disjoint subsets of $\cap_j U_j$. We declare that U_0 holds for every element. We declare that each V_k holds on at least one element. Lastly, we set V_0 to be active.

Stage s : If W_F does not enumerate any number at stage s , we declare that U_s is the same as U_{s-1} and that R_s holds on no tuple.

If W_F enumerates some number at stage s , we declare that U_s is an infinite co-infinite subset of U_{s-1} , and that R_s defines a bijection between U_s and $U_{s-1} \setminus U_s$.

If W_I does not enumerate anything at stage s , we declare that each of V_k for $k \geq l$ where V_l is active contain at least s elements.

If W_I enumerates some number at stage s , we declare that the active V_l contains exactly s elements and declare V_{l+1} to be active.

This gives a recursively enumerable list of axioms for T , and we leave it to the reader to verify that this list of axioms determines a complete theory. If, in fact, W_I is infinite

and W_F is finite, then $U = \bigcap_i U_i$ is a strongly minimal formula, where U is definable since the intersection is truly finite. The V_k , despite there being countably many, all describe finite subsets of U . Further, since the R_l provide a one-to-finite correspondence between U and the universe of a model, U is a 1-cardinal strongly minimal set.

If, on the other hand, W_I is finite, then countably many V_k define disjoint infinite subsets of U . By taking \aleph_1 -sized models where each V_k except one is countable, we have many non-isomorphic \aleph_1 -sized models.

Similarly, if W_F is infinite, then T_{IF} has continuum many 1-types, so it is not uncountably categorical.

Thus, T_{IF} is uncountably categorical if and only if W_I is infinite and W_F is finite. \square

References

- [1] John T. Baldwin and Alistair H. Lachlan, *On strongly minimal sets*, J. Symbolic Logic **36** (1971), 79–96.
- [2] Wilfrid Hodges, *A Shorter Model Theory*, Cambridge University Press, Cambridge, 1997.
- [3] Steffen Lempp and Theodore A. Slaman, *The complexity of the index sets of \aleph_0 -categorical theories and of Ehrenfeucht theories*, Advances in Logic, Contemp. Math., vol. 425, Amer. Math. Soc., Providence, RI, 2007, pp. 43–47.
- [4] Robert I. Soare, *Recursively Enumerable Sets and Degrees*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1987