

THE DEGREES OF CATEGORICAL THEORIES WITH RECURSIVE MODELS

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ABSTRACT. We show that even for categorical theories, recursiveness of the models guarantees no information regarding the complexity of the theory. In particular, we show that every tt -degree reducible to $0^{(\omega)}$ contains both \aleph_1 -categorical and \aleph_0 -categorical theories in finite languages all of whose countable models have recursive presentations.

1. INTRODUCTION

A fundamental question of recursive model theory is to understand the relationship between the complexity of a theory and the complexity of presentations of its models. It is well known that if a theory is recursive, then the Henkin construction produces a decidable model, that is a model whose elementary diagram is recursive. Also, if T is recursive and \aleph_1 -categorical, then all of its countable models are decidable presentable [7][10]. On the other hand, if T has a recursive model, that is a model whose atomic diagram is recursive, then in general we can only say that T is tt -reducible to $0^{(\omega)}$. Naturally, one would like to know whether this bound can be improved upon for tame theories. Two natural classes of tame theories are the \aleph_0 -categorical and \aleph_1 -categorical theories.

Goncharov and Khousseinov [5] showed that for each n there is an \aleph_1 -categorical, non- \aleph_0 -categorical theory turing equivalent to $0^{(n)}$ all of whose countable models have recursive presentations. They also showed that for each n there is an \aleph_0 -categorical theory turing equivalent to $0^{(n)}$ with a recursive countable model. Fokina [4] extended these result from $0^{(n)}$ to any arithmetical turing degree. Goncharov and Khousseinov conclude by asking whether there is a theory turing equivalent to $0^{(\omega)}$ which is \aleph_1 -categorical and all of its countable models are recursive, and also whether there is an \aleph_0 -categorical theory turing equivalent to $0^{(\omega)}$ with a recursive countable model.

The latter question was answered by Khousseinov and Montalban [11] in the affirmative. They generalized the construction of the random graph to allow the theory to code true arithmetic. We will answer the former question also in the affirmative. To answer the question, we construct an \aleph_1 -categorical, in fact strongly minimal, theory in a finite language with no level of quantifier elimination even after any collection of parameters are named. This cannot be achieved with a disintegrated theory [6]. In fact, any disintegrated theory with a recursive model is

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turing-below $0''$. This bound also holds for modular strongly minimal expansions of groups [2], and trivially for field-like expansions of algebraically closed fields, which are all definitional expansions [14]. Thus the result cannot be achieved with the canonical examples of strongly minimal theories satisfying the Zilber trichotomy. Due to this, we construct the theory via a Hrushovski construction.

In fact, we will show the stronger result that if \mathbf{d} is any tt -degree reducible to $0^{(\omega)}$, then there exists a strongly minimal theory $T \in \mathbf{d}$ in a finite language all of whose models are (uniformly) recursively presentable.

There will be two components to the proof of this recursive model theoretic result. One, the infinite language version of the Hrushovski amalgamation construction [1] as presented in section 1.1, is a model theoretic tool for building strongly minimal theories. The other, the Ash-Knight metatheorem [3] as presented in section 1.2, is the recursion theoretic tool to manage $0^{(\omega)}$ -level information. First we will define the theory by the model theoretic construction, and then we will verify by use of the metatheorem that all of the countable models of the theory have recursive presentations.

In Section 4, we will similarly show that if \mathbf{d} is any tt -degree below $0^{(\omega)}$, then there exists an \aleph_0 -categorical theory $T \in \mathbf{d}$ in a finite language whose countable model is recursively presentable. The proof follows very similar lines, where we replace the Hrushovski amalgamation by a Fraïssé amalgamation and we replace the metatheorem argument with a theorem of Knight.

1.1. Infinite language Hrushovski constructions. This section is a summary of sections 2 and 3 of [1], which in turn is an adaptation of Hrushovski's original construction [9] to an infinite language. There is no new content in this section, and it is included for self-containment of this paper. The key concept of a Hrushovski construction is to fix a function δ on finite L -structures which will serve as an approximation to their dimension. The true dimension of any finite set A will be $d(A) = \min\{\delta(B) \mid B \supseteq A\}$. We will then form a model by amalgamating finite L -structures as much as possible while ensuring that no finite set has negative dimension and that any extension of dimension 0 has at most a fixed finite number of copies.

Let L be a countable relational language. For ease of notation, we assume L contains only ternary relation symbols, say with signature $\{R_i \mid i \in I\}$. In the structure we generate, we ensure that each relation symbol is symmetric and holds only on distinct tuples. For a finite L -structure A , we write $|R(A)|$ for the number of tuples from A on which R holds. The following definitions are standard to Hrushovski constructions, and we will use them throughout.

Definition 1. *For any finite L -structures A and B and infinite L -structure D , we define:*

- $\delta(A) = |A| - \sum_{i \in I} |R_i(A)|$.
- $\delta(B/A) = \delta(A \cup B) - \delta(A)$.
- If $A \subseteq B$, we set $\delta(A, B) = \min\{\delta(C) \mid A \subseteq C \subseteq B\}$.
- Similarly, if $A \subseteq D$, we set $\delta(A, D) = \min\{\delta(C) \mid A \subseteq C \subset D, C \text{ finite}\}$.
- If $A \subseteq B$, we say A is strong in B or $A \leq B$ if $\delta(A) = \delta(A, B)$.
We say A is strong in D if $A \subseteq D$ and A is strong in C for each finite $A \subseteq C \subset D$.

- We say B is simply algebraic over A if $A \cap B = \emptyset$, $A \leq A \cup B$, $\delta(B/A) = 0$, and there is no proper subset B' of B such that $\delta(B'/A) = 0$.
- We say that B is minimally simply algebraic over A if B is simply algebraic over A and there is no proper subset A' of A such that B is simply algebraic over A' .
- We say A and B are freely joined if $\delta(A \cup B) = \delta(A) + \delta(B) - \delta(A \cap B)$.

Note that A and B are freely joined if and only if all relations on $A \cup B$ are on tuples entirely from A or entirely from B . The following are key lemmas which verify that δ and \leq act as expected.

Lemma 2. (Submodularity Property) For any finite L -structures $A, B \subseteq C$, $\delta(A \cup B) \leq \delta(A) + \delta(B) - \delta(A \cap B)$.

Lemma 3. Let A be a finite L -substructure of N . Suppose $A \leq N$.

- (1) $\delta(X \cap A) \leq \delta(X)$ whenever $X \subseteq N$ is finite.
- (2) $\delta(A', A) = \delta(A', N)$ whenever $A' \subseteq A$.
- (3) In particular, if $A' \leq A \leq N$, then $A' \leq N$.

Lemma 4. If X, A , and B are finite L -structures such that $A \subseteq B$, then $\delta(X/A \cup (X \cap B)) \geq \delta(X/B)$. In particular, if $X \cap B = \emptyset$, then $\delta(X/A) \geq \delta(X/B)$.

Up to this point, the definitions and lemmas have all been as in [9]. The following definition is necessary to allow for the infinite language. We need the condition that we bound the number of extensions of relative dimension 0 to be first order. Since the language is infinite, it would not be first order to bound the number of extensions $Y \supset X$ isomorphic to a particular pair $B \supset A$. Instead, we bound the number of extensions Y over X enough like $B \supset A$ that the relations showing $\delta(B/A) = 0$ suffice to witness that $\delta(Y/X) \leq 0$. This idea is the content of the next definition.

Definition 5. • For any disjoint L -structures A and B , we write $tp_{r.q.f.}(B/A)$ for the set $\{R_i(\bar{x}) \mid \bar{x} \subseteq (B \cup A)^3 \setminus A^3, i \in I, \text{ and } R_i(\bar{x}) \text{ holds}\}$. We call this set the relative quantifier-free type of B over A .

- Let $L_{B/A}$ be the language generated by $\{R_i \mid \exists \bar{x} \in (B \cup A)^3 \setminus A^3 (R_i(\bar{x}))\}$, i.e., the language appearing in $tp_{r.q.f.}(B/A)$.
- Suppose Y and X are finite L -structures such that Y is minimally simply algebraic over X and that B and A are finite L -structures such that $tp_{r.q.f.}(B/A)|_{L_{Y/X}} = tp_{r.q.f.}(Y/X)$ and $tp_{q.f.}(A) = tp_{q.f.}(X)$. In other words, A is isomorphic to X and up to the language $L_{Y/X}$, B/A is isomorphic to Y/X . Then we say the extension B over A is of the form of Y over X .

We will define the amalgamation class with the two conditions that all finite sets have non-negative dimension and that a uniform bound holds on the number of extensions over a set A of the form of Y over X . To do so, we fix a bound function μ with the following properties:

- μ is a function from pairs of L -structures to ω such that $\mu(Y, X)$ depends only on the atomic type of the pair (Y, X) .
- For any Y and X , $\mu(Y, X) \geq |X|$.
- If Γ is a relative quantifier-free type, there exists a sublanguage L' of L with finite subsignature such that whenever $tp_{r.q.f.}(Y/X) = \Gamma = tp_{r.q.f.}(Y'/X')$ and $tp_{q.f.}(X)|_{L'} = tp_{q.f.}(X')|_{L'}$, then $\mu(Y, X) = \mu(Y', X')$.

The first condition is necessary for μ to be well defined, the second condition is necessary for the combinatorics of lemma 8, and the last condition is again necessary to ensure that μ is a first-order bound.

Definition 6. Let $\mathcal{C} = \mathcal{C}_\mu$ be the class of finite L -structures C such that the following hold:

- If $A \subseteq C$ then $\delta(A) \geq 0$.
- Let Y/X be a minimally simply algebraic extension. Let B_1, \dots, B_n, A be disjoint subsets of C such that B_i/A is of the form of Y/X for each i . Then $n \leq \mu(Y, X)$.

To generate a model from \mathcal{C} , we verify that \mathcal{C} has a strong amalgamation property. To see this, some combinatorics must be done. The key to the combinatorics is the following lemma, which will be used again in section 3.

Lemma 7. Let A, B_1, B_2 be L -structures such that any substructure has non-negative δ -value, $A = B_1 \cap B_2$, and $A \leq B_1$. Let E be the free-join of B_1 with B_2 over A . Suppose C^1, \dots, C^r, F are disjoint substructures of E such that each C^i is minimally simply algebraic over F and the structures C^i and C^j are isomorphic over F for each $1 \leq i, j \leq r$. Then one of the following holds:

- (1) One of the C^i is contained in $B_1 \setminus A$ and $F \subseteq A$.
- (2) Either $F \cup \bigcup_{i=1}^r C^i \subseteq B_2$ or both $F \cup \bigcup_{i=1}^r C^i \subseteq B_1$ and one of the C^i is contained in $B_1 \setminus A$.
- (3) $r \leq \delta(F)$
- (4) For one C^j , setting $X = (F \cap A) \cup (C^j \cap B_2)$, $\delta(X/X \cap A) < 0$. Further, one of the C^j is contained in $B_1 \setminus A$. (Note that this cannot happen if $A \leq B_2$ by Lemma 3).

Proof. A careful reading of Lemma 3 of [9] will show that this is indeed what is proved there. \square

Lemma 8. (Strong Amalgamation Lemma) Suppose $A, B, C \in \mathcal{C}$, $A \leq B$, $A \leq C$. Then there exists $D \in \mathcal{C}$ such that $C \leq D$, and an embedding $g : B \rightarrow D$ such that $g(B) \leq D$ and $g(A) = id|_A$.

Proof. We refer the reader to Lemma 13 of [1]. \square

The following theorem summarizes the results of section 3 of [1].

Theorem 9. Amalgamation of \mathcal{C} yields a unique countable generic amalgam \mathcal{M} . This \mathcal{M} is saturated and strongly minimal. For $a \in \mathcal{M}$ and finite $B \subset \mathcal{M}$, $a \in \text{acl}(B)$ if and only if $d(aB) = d(B)$. Thus the algebraic dimension of a finite $B \subset \mathcal{M}$ is $d(B)$. (Recall that $d(X) = \min\{\delta(Y) \mid Y \supseteq X, Y \text{ finite}\}$.)

1.2. The Ash-Knight metatheorem. This section entirely follows Ash-Knight [3] (see pg. 236), with the notational exception that the set which we call V is there referred to as L . To maintain as much of the Ash-Knight notation as possible, we still refer to elements of V as l 's.

Let V and U be recursively enumerable sets, E be a partial recursive enumeration function on V , and let P be a recursively enumerable alternating tree on V and U made up of non-empty finite sequences which all start with the same $\hat{l} \in V$. Let $(\leq_n)_{n \in \omega}$ be uniformly recursively enumerable binary relations on V .

We define the tuple $(V, U, \hat{l}, P, E, (\leq_n)_{n < \omega})$ to be an ω -system if it satisfies the following properties:

- (1) \leq_n is reflexive and transitive for all $n < \omega$.
- (2) $l \leq_n l' \Rightarrow l \leq_m l'$ for $m < n < \omega$.
- (3) If $l \leq_0 l'$, then $E(l) \subseteq E(l')$.
- (4) If $\tau \hat{\ } u \in P$, where τ has length $2n + 1$ ending in $l^0 \in V$ and

$$l^0 \leq_{n_0} l^1 \leq_{n_1} \dots \leq_{n_{t-1}} l^t$$

for $n > n_0 > \dots > n_t$, then there exists l^* such that $\tau ul^* \in P$ and $l^i \leq_{n_i} l^*$ for each $0 \leq i \leq t$.

Theorem 10. (*Ash-Knight Metatheorem*, [3] *Thm 14.4 with $\alpha = \omega$*)

Let $(V, U, \hat{l}, P, E, (\leq_n)_{n < \omega})$ be an ω -system, and let q be a uniformly $0^{(n)}$ instruction function for P (ie: uniformly in n , q computes $u_n \in U$ from input $\hat{l}u_0l_0u_1l_1 \dots l_{n-1}$ using oracle 0^n). Then there is a path $\pi = \hat{l}u_0l_0u_1l_1 \dots$ through P which agrees with the instruction function q such that $E(\pi)$ is recursively enumerable.

We will use the metatheorem to construct structures whose atomic diagrams are recursively enumerable, i.e., recursive structures. Note that the proof that a particular ω -system satisfies property 4 contains all the details as to how injury is handled in the construction.

2. OUR THEORY

Using the construction outlined in section 1.1, we need only specify a language L and a function μ to produce a strongly minimal theory $\text{Th}(\mathcal{M})$. The theory we seek will be a reduct of this one. We specify $L = \{R_i | i \in \omega\}$ with each symbol ternary.

The function μ will be our tool to encode recursion theoretic information into our theory. Given any tt -degree $\mathbf{d} \leq_{tt} \mathbf{0}^{(\omega)}$, fix a set $S \in \mathbf{d}$ with the property that whether i is in S is uniformly answered by $0^{(i)}$. Every tt -degree below $0^{(\omega)}$ contains such a set. We want to build a theory T so that $T \equiv_{tt} S$, so we will encode whether $i \in S$ into the bound for some minimally simply algebraic extension.

We fix a particular minimally simply algebraic extension H over a three element set G such that $tp_{r,q,f.}(H/G)$ involves a single ternary relation symbol. We fix $k = |G \cup H|$. For $i \geq 1$, we write Γ_i for the relative quantifier-free type formed by replacing the relation symbol in $tp_{r,q,f.}(H/G)$ by R_{i-1} . If B and A are finite L -structures such that $tp_{r,q,f.}(B/A)|_{R_{i-1}} = \Gamma_i$, then we say the extension B over A is a Γ_i -extension. We want to code whether or not $i \in S$ into the number of allowed Γ_i -extensions. This is the motivation for the difference between lines 3 and 4 of the following definition. The difference of Line 2 with that of 3 or 4 and the appearance of 2^k throughout is to ensure that each R_i is definable in terms of R_{i-1} as in Lemma 13.

Definition 11. Let B be minimally simply algebraic over A .

$$\mu(B, A) = \begin{cases} |A| + 2^k & \text{if } B \text{ over } A \text{ is not a } \Gamma_i\text{-extension for any } i \\ |A| + 2^k + 2 & \text{if } B \text{ over } A \text{ is a } \Gamma_i\text{-extension and } \neg R_i(A) \\ |A| + 2^k + 1 & \text{if } B \text{ over } A \text{ is a } \Gamma_i\text{-extension, } R_i(A), \text{ and } i \in S \\ |A| + 2^k & \text{if } B \text{ over } A \text{ is a } \Gamma_i\text{-extension, } R_i(A), \text{ and } i \notin S \end{cases}$$

We fix $T^+ = \text{Th}(\mathcal{M})$ to be the strongly minimal L -theory of the amalgam of \mathcal{C}_μ as in section 1.1. Our immediate task is to show that in T^+ each R_i is definable in

terms of R_{i-1} and that T^+ encodes S . To do this, we need the following form of the algebraic amalgamation lemma specifically for Γ_i . It is for the combinatorics in the proof of this lemma that the 2^k term appears in μ .

Lemma 12. (*Algebraic Amalgamation Lemma for Γ_i*) *Suppose $A, B_1, B_2 \in \mathcal{C}$, $A = B_1 \cap B_2$, $B_1 \setminus A$ is simply algebraic over A . Suppose further that $B_1 \setminus A$ is minimally simply algebraic over $A' \subseteq A$, and $\text{tp}_{r.q.f.}(B_1 \setminus A/A') = \Gamma_i$. Then the free-join of B_1 with B_2 over A is a member of \mathcal{C} unless B_2 contains $\mu(B_1 \setminus A, A')$ disjoint Γ_i -extensions over A' .*

Proof. Suppose the free-join E is not in \mathcal{C} . This means that there are C^1, \dots, C^n, F contained in E and a pair (Y, X) such that each C^j over F is of the form of Y over X and $n > \mu(Y, X)$. Restricting E to the language $L_{Y/X}$, we see each of the C^j are minimally simply algebraic over F in the same way. Here we have the same set-up as in Hrushovski's algebraic amalgamation lemma ([9], Lemma 3). Claims 1-3 and case 1 of Hrushovski's algebraic amalgamation lemma hold exactly as proved there. Case 1 leads to the exception in this lemma. In case 2, we need only count the number of C^j which are entirely contained in $B_1 \setminus A$. There are certainly fewer than $2^{|(B_1 \setminus A) \cup A'|} = 2^k$ such C^j . Thus, $n \leq |F| + 2^k \leq \mu(Y, X)$. \square

We will write $\Gamma_i(\bar{y}, \bar{x})$ to denote the first order formula designating that \bar{y} over \bar{x} is a Γ_i -extension. Note that the formula $\Gamma_i(\bar{y}, \bar{x})$ involves only the relation symbol R_{i-1} .

Lemma 13. *For every $i \geq 1$, $\mathcal{M} \models R_i(\bar{x}) \leftrightarrow \neg \exists^{5+2^k} \bar{y} (\Gamma_i(\bar{y}, \bar{x}))$. (We write $\exists^m \bar{y} \phi(\bar{y})$ to represent the formula stating that there exists m disjoint tuples \bar{y} satisfying ϕ .)*

Proof. The rightward direction follows from the fact that any finite substructure of \mathcal{M} is an element of \mathcal{C} . If the rightward direction did not hold, then we would be violating the μ -bound for Γ_i -extensions.

The leftward direction follows from the previous lemma. Suppose $\neg R_i(\bar{x})$ holds. Let B be such that $\bar{x} \subseteq B \leq \mathcal{M}$. Repeated application of the previous lemma amalgamating a Γ_i -extension with B over \bar{x} shows that there is a $C \in \mathcal{C}$ such that $B \leq C$ and C contains $5 + 2^k$ disjoint Γ_i -extensions over \bar{x} . The fullness of the amalgamation of \mathcal{M} (i.e., whenever $B \leq M$ and $B \leq C$, there is an embedding $f : C \rightarrow M$ so that $f(C) \leq M$ and $f|_B = \text{id}_B$. This is property 3 in [1].) guarantees that this C embeds in \mathcal{M} over B . \square

Since the first order formula $\Gamma_i(\bar{y}, \bar{x})$ is defined using only the relation R_{i-1} , we see that each of the R_i are definable via the relation R_0 .

Definition 14. *Let $T = T^+|_{R_0}$.*

By Lemma 13, T^+ is a definitional expansion of T . This T is going to be the theory referenced in our main theorem. Note that the defining formulae for R_i from R_0 do not depend on S and are uniformly recursive, so $T \equiv_{tt} T^+$. The following lemma shows that T^+ , and thus T , tt -computes S .

Lemma 15. *For every $i \geq 1$, $i \in S$ if and only if $\mathcal{M} \models \forall \bar{x} \exists^{4+2^k} \bar{y} (\Gamma_i(\bar{y}, \bar{x}))$.*

Proof. There are as many disjoint Γ_i -extensions in \mathcal{M} over \bar{x} as μ allows, by the same argument as in Lemma 13. If $i \in S$, then μ always allows at least $4 + 2^k$

extensions. If $i \notin S$ and $R_i(\bar{x})$, then μ allows only $3 + 2^k$ Γ_i -extensions over \bar{x} , witnessing that $\mathcal{M} \models \neg \forall \bar{x} \exists^{4+2^k} \bar{y} (\Gamma_i(\bar{y}, \bar{x}))$. \square

This shows that $S \leq_{tt} T^+ \equiv_{tt} T$. T is also recursive in S (Properties 2 and 3'' from [1] are explicit axiom schemata for T which are recursive in \mathcal{C} , which in turn is recursive in S), and since the construction works for any S , this computation is total, yielding $S \equiv_{tt} T$. All that remains is to show that each countable model of T is recursive.

3. CONSTRUCTING THE COUNTABLE MODELS OF T

Before using the metatheorem to recursively construct models of T , we recall the construction of a model of T ignoring the recursion theoretic issues. The following will construct a k -dimensional model of T^+ , i.e., a k -dimensional model of T .

Stage 0: Let M_0 be the structure with k elements and no relations holding between them.

Stage $s > 0$: We are given M_{s-1} , the structure constructed at the previous stage. List off the first s minimally simply algebraic extensions over subsets of M_{s-1} : $B_1/A_1, \dots, B_s/A_s$. If the free amalgam of B_1 with M_{s-1} over A_1 is in \mathcal{C} , replace M_{s-1} with this amalgam. Repeating this procedure for $j \leq s$ gives a new structure, which we call M_s .

This gives a chain of structures $M_0 \leq M_1 \leq \dots M_k \dots \leq \cup_i M_i$, where $\cup_i M_i$ is the k -dimensional model of T^+ . We wish to employ this construction, but we cannot recursively compute μ , so we cannot recursively tell whether an amalgam is in \mathcal{C} . In fact, the k -dimensional model of T^+ will not be recursive. We will rather work with approximations to S , thus to μ and \mathcal{C} , and will build approximations to a model of T^+ , but we will injure our R_i -assignments when our approximation to the question of whether $i \in S$ changes. Of course, we will use the metatheorem to do this coherently for all i . Recall that whether $i \in S$ is uniformly answered by $0^{(i)}$.

We use the Ash-Knight Metatheorem to construct the k -dimensional model of T by defining an ω -system $(V, U, \hat{l}, P, E, (\leq_n)_{n \in \omega})$. Throughout the construction, we will be working with various estimates to the set S . These estimates will be represented by elements of $2^{<\omega}$. Given the estimate σ , we define μ_σ and \mathcal{C}_σ , the corresponding approximations to μ and \mathcal{C} .

Definition 16. Let σ be an element of $2^{<\omega}$ where $\text{length}(\sigma) = n+1$ ($\sigma(0)$ is never referenced, so this index is off by one).

We define $L_\sigma = L_n$ to be the language generated by the relation symbols $\{R_i | i < n\}$

For B a minimally simply algebraic extension over A , let

$$\mu_\sigma(B, A) = \begin{cases} |A| + 2^k & \text{if } B \text{ over } A \text{ is not a } \Gamma_i\text{-extension for any } i \in \omega \\ |A| + 2^k + 2 & \text{if } B \text{ over } A \text{ is a } \Gamma_i\text{-extension and } \neg R_i(A) \\ |A| + 2^k + 1 & \text{if } B \text{ over } A \text{ is a } \Gamma_i\text{-extension, } R_i(A), \text{ and } \sigma(i) = 1 \\ |A| + 2^k & \text{if } B \text{ over } A \text{ is a } \Gamma_i\text{-extension, } R_i(A), \text{ and } \sigma(i) = 0 \end{cases}$$

Note that $\mu_\sigma(B, A)$ is defined for any finite L_σ extension, so the following is well defined. Let \mathcal{C}_σ be the class of finite L_σ structures C such that the following conditions hold:

- $\delta(A) \geq 0$ for all $A \subseteq C$.
- Let Y over X be a minimally simply algebraic extension. Suppose B_1, \dots, B_n , A are disjoint subsets of C such that each B_i over A is an extension of the form of Y over X . Then $n \leq \mu_\sigma(Y, X)$.

Definition 17. Fix $k \in \omega$. We define S_k to be the following system (which will construct the k -dimensional model of T):

- V is the set of pairs (M, σ) where M is a finite L -structure whose universe is an initial segment of ω , $\sigma \in 2^{<\omega}$, and $M \in \mathcal{C}_\sigma$. For $l \in V$, we write $l = (M_l, \sigma_l)$.
- U is $2^{<\omega}$.
- \hat{l} is the pair (M, σ) where M is the structure with k elements and no relations and σ is the trivial string of length 0.
- $E(l)$ is the set of primitive (atomic and negations of atomic) statements true about M_l in the language generated by the single relation symbol R_0 .
- $l \leq_n l'$ if the following conditions hold:
 - (1) $\sigma_l|_n = \sigma_{l'}|_n$ (ie: $\sigma_l(i) = \sigma_{l'}(i)$ for $i \leq n$).
 - (2) The universe of M_l is a subset of the universe of $M_{l'}$.
 - (3) $M_l|_{L_n} \leq M_{l'}|_{L_n}$. By this we mean that as L_n -structures $M_l|_{L_n} \subseteq M_{l'}|_{L_n}$ and it is a strong substructure.
- P is the tree where $\hat{l}u_0l_0 \dots u_t l_t \in P$ if
 - (1) For each i , $\sigma_{l_i} = u_i$.
 - (2) For each i , $M_{l_i} \leq M_{l_{i+1}}$.
 - (3) For each i , $\delta(M_{l_i}) = k$.
 - (4) Take a universal list of all minimally simply algebraic extensions in L along with sets the extension could be over, call it *List*. For the first i entries on *List*, the free-join of the extension with M_{l_i} is not in $\mathcal{C}_{\sigma_{l_i}}$. (the last item says that all the copies of each of the first i extensions which are allowed in $\mathcal{C}_{\sigma_{l_i}}$ occur already in M_{l_i})

Theorem 18. S_k is an ω -system.

Proof. We focus on the difficult condition.

Suppose $\tau u \in P$, $\text{length}(\tau) = 2n+1$, τ ends in l^0 , and

$$l^0 \leq_{n_0} l^1 \leq_{n_1} \dots \leq_{n_{t-1}} l^t$$

for $n > n_0 > n_1 > \dots > n_{t-1} > n_t$. Without loss of generality, we assume that $n_0 = n - 1$. We need to show that there exists an l^* such that $\tau u l^* \in P$, and for each i , $l^i \leq_{n_i} l^*$. First we will define an auxiliary structure $l^\# = (\mathcal{N}, \sigma)$ which will handle the injury occurring in this sequence of l 's. To form \mathcal{N} , we keep the occurrences of relation symbols from the M_{l_j} to which we are committed, and forget all of the others. Some combinatorics will be required to verify that $\mathcal{N} \in \mathcal{C}_u$. We will then extend $l^\#$ to an l^* which has the right dimension and contains amalgamations of the required minimally simply algebraic extensions from \mathcal{C}_u . To avoid notation such as M_{l_j} , we write $l^j = (M_j, \sigma_j)$.

Let $l^\#$ be the pair (\mathcal{N}, σ) , defined as follows: $\sigma = u$ and \mathcal{N} has the same universe as M_t . Let \bar{x} be a tuple in \mathcal{N} . We now describe whether or not R_i holds on \bar{x} . Let m be least such that $\bar{x} \subseteq M_m$. Then R_i holds on \bar{x} in \mathcal{N} if and only if R_i is in L_{n_m} and holds on \bar{x} in M_m .

We will write \mathcal{N}_i for the substructure of \mathcal{N} with the same universe as M_i , and we will write \mathbb{L}_i for L_{n_i} .

Claim 19. $M_i|_{\mathbb{L}_i} = \mathcal{N}_i|_{\mathbb{L}_i}$, i.e., for each relation in L_i and every tuple in M_i , the relation holds in M_i if and only if it holds in \mathcal{N} .

Proof. For each $j \leq i$, $l^j \leq_{n_{i-1}} l^i$. In particular, $l^j \leq_{n_i} l^i$. Let \bar{x} be any tuple in M_i and let m be minimal such that $\bar{x} \subseteq M_m$. Then $m \leq i$, so $l^m \leq_{n_i} l^i$. Thus for $R \in \mathbb{L}_i$, $R(\bar{x})$ holds in M_m if and only if it holds in \mathcal{N} (by definition of \mathcal{N}), and the first condition is equivalent to $R(\bar{x})$ holding in M_i , since $l^m \leq_{n_i} l^i$. \square

In particular, $\mathcal{N}_0 = M_0$ since $n_0 = n - 1$.

Lemma 20. $l^\# \in V$

Proof. We need to verify that $\mathcal{N} \in \mathcal{C}_\sigma$. We verify this by verifying each condition in the definition of \mathcal{C}_σ .

- (1) $\delta(A) \geq 0$ for all $A \subseteq \mathcal{N}$

Proof. Let A be a subset of \mathcal{N} . Let $A_i = A \cap \mathcal{N}_i$. We need to show that $\delta(A) \geq 0$. We achieve this by showing that $\delta(A_{i+1}/A_i) \geq 0$ for each i . This suffices since $\delta(A_t) = \delta(A_t/A_{t-1}) + \delta(A_{t-1}/A_{t-2}) + \dots + \delta(A_1/A_0) + \delta(A_0)$ and A_0 is a subset of $\mathcal{N}_0 = M_0$, hence has non-negative δ -value.

$\delta(A_{i+1}/A_i)$ is $|A_{i+1} \setminus A_i| -$ (the number of relations holding in A_{i+1} involving at least one element in $A_{i+1} \setminus A_i$). Consider B the subset of M_{i+1} with the same underlying set as A_{i+1} . Since $M_i|_{\mathbb{L}_i} \leq M_{i+1}|_{\mathbb{L}_i}$, $\delta(B|_{\mathbb{L}_i}/(B \cap M_i)|_{\mathbb{L}_i}) \geq 0$, but $\delta(B|_{\mathbb{L}_i}/(B \cap M_i)|_{L_i}) \leq \delta(A_{i+1}/A_i)$ as every relation counting on the right counts on the left as well. Thus, each summand is non-negative and $\delta(A) \geq 0$. \square

- (2) If C^1, \dots, C^n, F are disjoint subsets of \mathcal{N} , and each C^j over F is of the form of Y over X , a minimally simply algebraic extension, then $n \leq \mu_\sigma(Y, X)$.

Proof. We proceed by induction to show that the condition holds for each \mathcal{N}_i . The condition holds on \mathcal{N}_0 , as this is just M_0 and μ_σ agrees with μ_{σ_0} . Supposing the condition holds for \mathcal{N}_{s-1} , we will show that the condition holds on \mathcal{N}_s as well.

The proof follows via Lemma 7. Suppose C^1, \dots, C^n, F are disjoint subsets of \mathcal{N}_s , and each C^j over F is of the form of Y over X . Since $\mathcal{N}_{s-1} \leq \mathcal{N}_s$, we apply Lemma 7 with $B_1 = \mathcal{N}_s|_{L_{Y/X}}$, $A = B_2 = \mathcal{N}_{s-1}|_{L_{Y/X}}$. There are 4 cases to consider. In one case, $r \leq |X| < \mu_\sigma(Y, X)$. In each of the other cases, one C^j is entirely contained in $\mathcal{N}_s \setminus \mathcal{N}_{s-1}$.

In the case that one C^j is entirely contained in $\mathcal{N}_s \setminus \mathcal{N}_{s-1}$, $tp_{r.q.f.}(Y/X)$ only includes relations from the language \mathbb{L}_s . There are a number of possibilities to consider:

case 1: Y over X is not a Γ_i -extension for any i .

Let F_0 be the subset of M_s with underlying set the same as F . Since $M_s|_{\mathbb{L}_s} = \mathcal{N}_s|_{\mathbb{L}_s}$, we see that each of the C^j over F , looked at as subsets of M_s , are of the form of Y over X' , where $X' \cong F_0$. Since for non- Γ_i -extensions, $\mu_\sigma(Y, X)$ only depends on $|X|$, and since M_s satisfies the property for μ_{σ_s} , we have $n \leq \mu_{\sigma_s}(Y, X') = \mu_\sigma(Y, X)$.

case 2: Y over X is a Γ_i -extension and $\neg R_i(X)$.

In this case, we also look at the C^j and F as subsets of M_s . The C^j are each Γ_i -extensions over F . If $R_i(F)$ is in M_s , then the number of C^j is bounded by $\mu_{\sigma_s}(Y, X')$ (where X' is the same as X but with $R_i(X)$ holding) which is even less than $\mu_{\sigma_s}(Y, X)$. If $\neg R_i(F)$ in M_s , then the number of C^j is bounded by $\mu_{\sigma_s}(Y, X)$. Since $\sigma_s|_{n_{s-1}} = \sigma|_{n_{s-1}}$, $\mu_{\sigma_s}(Y, X) = \mu_{\sigma}(Y, X)$.

case 3: Y over X is a Γ_i -extension and $R_i(X)$.

If $R_i \in \mathbf{L}_s$, then we have the corresponding fact in M_s , so we get the μ -bound from the fact that M_s satisfies the property for μ_{σ_s} and $\sigma_s|_{n_{s-1}} = \sigma|_{n_{s-1}}$. Now suppose $R_i \notin \mathbf{L}_s$. Since Y over X is a Γ_i -extension, $R_{i-1} \in \mathbf{L}_s$. So $R_i \in \mathbf{L}_{s-1}$. Clearly, $R_i(F)$ implies that $F \subseteq \mathcal{N}_{s-1}$. But then $R_i(F)$ holds in M_{s-1} , and $l^{s-1} \leq_{n_{s-1}} l^s$, so $R_i(F)$ holds in M_s as well. Again, we get the μ -bound from M_s .

This concludes the inductive step, showing that $\mathcal{N} = \mathcal{N}_t$ satisfies the condition. \square

\square

Claim 21. For each i , $l^i \leq_{n_i} l^\#$.

Proof. We verify the two properties. First we verify $\sigma_i|_{n_i} = \sigma|_{n_i}$. We know that $l^0 \leq_{n_{i-1}} l^i$, so $\sigma|_{n_i} = \sigma_0|_{n_i} = \sigma_i|_{n_i}$.

Second we verify that $M_i|_{\mathbf{L}_i} \leq \mathcal{N}|_{\mathbf{L}_i}$. Claim 19 gives us that $M_i|_{\mathbf{L}_i} \subseteq \mathcal{N}|_{\mathbf{L}_i}$.

Now, let X be a subset of $\mathcal{N}|_{\mathbf{L}_i}$. We use the same argument as before (when we showed that $\delta(A) \geq 0$ for all $A \subseteq \mathcal{N}$). We need to show that $\delta(X/\mathcal{N}_i|_{\mathbf{L}_i}) \geq 0$. For $i \leq j \leq t$, we write $X_j = ((X \cap \mathcal{N}_j) \cup \mathcal{N}_j)|_{\mathbf{L}_i}$. Then we have $\delta(X/\mathcal{N}_i|_{\mathbf{L}_i}) = \delta(X_t/X_{t-1}) + \delta(X_{t-1}/X_{t-2}) + \dots + \delta(X_{i+1}/X_i)$. As in the previous argument, each summand is non-negative, so $\delta(X/\mathcal{N}_i|_{\mathbf{L}_i}) \geq 0$. \square

The only obstructions to $l^\#$ being what we need for l^* is that it might not contain enough copies of the first n minimally simply algebraic extensions, and perhaps $\delta(\mathcal{N}) > k$. Extend \mathcal{N} using only the thus far unused relation symbol R_{n-1} to \mathcal{N}' so that $\delta(\mathcal{N}') = k$, $M_0 \leq \mathcal{N}'$, and $\mathcal{N}' \in \mathcal{C}_\sigma$. Then proceed to extend \mathcal{N}' to \mathcal{N}^* by amalgamating as many copies as allowed in \mathcal{C}_σ of the first n minimally simply algebraic extensions over \mathcal{N}^* . We set l^* to be (\mathcal{N}^*, u) . By construction, $l^i \leq_{n_i} l^\# \leq_{n_i} l^*$ and $\tau u l^* \in P$ as $l^* \in V$, $M_0 \leq \mathcal{N}^*$, $\sigma_{l^*} = u$, and $\delta(\mathcal{N}^*) = k$. Having found this l^* , we have shown that $(V, U, \hat{l}, E, P, (\leq_n)_{n \in \omega})$ is an ω -system. \square

We have a uniformly $0^{(n)}$ instruction function for u , namely $u_n = S|_n$, the string in $2^{<\omega}$ describing membership in S for integers $\leq n$. Thus, the metatheorem gives us a run, i.e., a path $\pi = \hat{l}u_0l_0u_1l_1\dots$ through P consistent with this instruction function such that $E(\pi)$ is recursively enumerable. $E(\pi)$ gives us the R_0 -atomic diagram of $\bigcup_i M_{l_i}$.

Theorem 22. The k -dimensional model of T is recursively presentable.

Proof. It remains to show that the model $M = \bigcup_i M_{l_i}$ is isomorphic to the k -dimensional model of T .

By construction, M satisfies properties 2 and 3'' of [1], thus $M \models T$. Furthermore, $\hat{l} \leq M$, so the dimension is at least k , and any finite $A \subseteq M$ is contained in some M_{l_i} , so has dimension $\leq \delta(M_{l_i}) = k$. Thus the dimension of M is k . \square

Theorem 23. The saturated model of T is recursively presentable.

Proof. We use a similar ω -system where instead of insisting that $\delta(M_{l_i}) = k$ in the definition of P , we insist that $\delta(M_{l_i}) \geq i$. To get \mathcal{N}' from \mathcal{N} we add a new element not related to anything else. In the final model, we constructed it so that 1, 2, 3', 3'' of [1] all hold so the constructed model is the saturated model of T . \square

Putting together Lemma 15, Theorem 22 and Theorem 23, we conclude the main theorem.

Theorem 24. *Let \mathbf{d} be a tt -degree reducible to $0^{(\omega)}$. There exists a strongly minimal theory $T \in \mathbf{d}$ in a finite language such that each countable model of T is recursively presentable.*

Proof. We have chosen $S \in \mathbf{d}$ and constructed $T \equiv_{tt} S$ so that all countable models of T have recursive presentations. \square

Corollary 25. *There exists a strongly minimal theory T in a finite language so that $T \equiv_T 0^{(\omega)}$ and each countable model of T is recursively presentable.*

4. THE \aleph_0 -CATEGORICAL CASE

As above, we fix a set S such that whether $n \in S$ is uniformly recursive in $0^{(n)}$. We work at first in an infinite language to produce the theory, then we will take a reduct to a sublanguage with finite subsignature. Let L be the language with signature $\{P, Q\} \cup \{R_i | i \geq 3\}$, where P and Q are binary relation symbols and each R_i is i -ary. We will generate an \aleph_0 -categorical theory via a Fraïssé construction (see [8], section 6.1) using the following amalgamation class \mathcal{K} .

Definition 26. *Let \mathcal{K} be the class of finite L -structures C which satisfy the following properties:*

- *Each relation symbol is symmetric and holds only on tuples of distinct elements.*
- *If $i - 10 \notin S$ or $i < 10$, then C satisfies*

$$\neg \exists \bar{x}, y, z \left[R_i(\bar{x}) \wedge P(y, z) \wedge \bigwedge_{\bar{w} \subset \bar{x}, |\bar{w}|=i-2} (R_{i-1}(y, \bar{w}) \wedge R_{i-1}(z, \bar{w})) \right]$$

- *If $i - 10 \in S$, then C satisfies*

$$\neg \exists \bar{x}, y, z \left[R_i(\bar{x}) \wedge Q(y, z) \wedge \bigwedge_{\bar{w} \subset \bar{x}, |\bar{w}|=i-2} (R_{i-1}(y, \bar{w}) \wedge R_{i-1}(z, \bar{w})) \right]$$

To carry out a Fraïssé construction using \mathcal{K} , we must verify the following lemma:

Lemma 27. *\mathcal{K} satisfies HP, JEP, and AP.*

Proof. Since \mathcal{K} is defined via \forall_1 formulae, it satisfies HP.

Given two disjoint structures $B, C \in \mathcal{K}$, see that the free-join of B with C over \emptyset is in \mathcal{K} , so \mathcal{K} satisfies JEP.

Lastly, given three structures $A, B, C \in \mathcal{K}$ where $A = B \cap C$, we will show that the free-join of B with C over A is in \mathcal{K} , and thus \mathcal{K} satisfies AP. Suppose the free-join is not in \mathcal{K} . Then there exists i, \bar{x}, y, z witnessing this. We may assume $i - 10 \in S$, so \bar{x}, y, z satisfy:

$$R_i(\bar{x}) \wedge Q(y, z) \wedge \bigwedge_{\bar{w} \subset \bar{x}, |\bar{w}|=i-2} (R_{i-1}(y, \bar{w}) \wedge R_{i-1}(z, \bar{w}))$$

Since any two elements of $\bar{x} \cup \{y, z\}$ are related by some relation, the whole set is contained in either B or C . Thus either $B \notin \mathcal{K}$ or $C \notin \mathcal{K}$, yielding a contradiction. \square

Now we use Fraïssé's theorem ([8], Theorem 6.1.2) which guarantees a countable ultra-homogeneous L -structure \mathcal{M} with $\text{Age}(\mathcal{M}) = \mathcal{K}$. As \mathcal{M} is ultra-homogeneous, it admits quantifier elimination. Thus, the number of n -types is bounded by the number of possible configurations of the finitely many relations in L of arity $\leq n$, showing that $\text{Th}(\mathcal{M})$ is \aleph_0 -categorical. The following lemma allows the reduction to a finite sub-language.

Lemma 28. *Suppose $i > 3$:
If $i - 10 \notin S$ or $i < 10$, then*

$$\mathcal{M} \models R_i(\bar{x}) \leftrightarrow \neg \exists y, z \left[P(y, z) \wedge \bigwedge_{\bar{w} \subset \bar{x}, |\bar{w}|=i-2} (R_{i-1}(y, \bar{w}) \wedge R_{i-1}(z, \bar{w})) \right]$$

Similarly, if $i - 10 \in S$, then

$$\mathcal{M} \models R_i(\bar{x}) \leftrightarrow \neg \exists y, z \left[Q(y, z) \wedge \bigwedge_{\bar{w} \subset \bar{x}, |\bar{w}|=i-2} (R_{i-1}(y, \bar{w}) \wedge R_{i-1}(z, \bar{w})) \right]$$

Proof. The rightward direction follows via the fact that $\text{Age}(\mathcal{M}) = \mathcal{K}$. To show the leftward direction, take a tuple \bar{x} such that $\mathcal{M} \models \neg R_i(\bar{x})$. By ultra-homogeneity of \mathcal{M} , it suffices to show that \bar{x} embeds into an element of \mathcal{K} where such a y and z exist. Consider the structure $A = \bar{x} \cup \{a, b\}$, where each of a and b are R_{i-1} -related to every $(i-2)$ -element subset of \bar{x} , a and b are P -related (Q -related in the case of the second equivalence above), and no other relations hold involving a or b . It is easy to verify that this structure is in \mathcal{K} . \square

Definition 29. *Let $T = \text{Th}(\mathcal{M})|_{\{P, Q, R_3\}}$*

By the previous lemma, $\text{Th}(\mathcal{M})$ is a definitional expansion of T . Further, from T we can in a *tt* way determine which definition of R_i is correct, and thus whether $i - 10 \in S$. This shows that $T \geq_{tt} S$. Further, by ultra-homogeneity of T , $T \leq_{tt} \mathcal{K} \leq_{tt} S$. It remains only to prove that the countable model of T is recursively presentable.

Lemma 30. *Uniformly in n , $T \cap \exists_n$ is computable in 0^{n-7} .*

Proof. We first show that any \exists_n formula in T is equivalent to a quantifier-free formula in the relations $\{P, Q, R_i\}_{i \leq 3+n}$. The proof proceeds by setting up the appropriate Ehrenfeucht-Fraïssé game (see: [13], section 2.4) and seeing that ‘ \exists loise’ (the second player) has a winning strategy. The game is the standard Ehrenfeucht-Fraïssé game of length n where we start with tuples \bar{a} and \bar{b} which have the same $\{P, Q, R_i\}_{i \leq 3+n}$ -quantifier-free type. Then whichever tuple \bar{c} ‘ \forall belard’ chooses, \exists loise can choose a tuple \bar{d} so that $\bar{a}\bar{c}$ and $\bar{b}\bar{d}$ satisfy the same $\{P, Q, R_i\}_{i \leq 3+n-1}$ -quantifier-free type. Proceeding as such, \exists loise wins the game of length n .

This shows that any \exists_n formula depends only on the relations $\{P, Q, R_i\}_{i \leq 3+n}$ ([13], Lemma 2.4.9). Thus, the \exists_n formula $\exists \forall \dots \phi(\bar{x})$ is equivalent to:

$$\bigvee_{(\text{configurations in } \{P, Q, R_i\}_{i \leq 3+n} \text{ in } \mathcal{K})} \bigwedge_{(\text{configurations in } \{P, Q, R_i\}_{i \leq 3+n-1} \text{ in } \mathcal{K})} \dots \phi(\bar{x})$$

To verify whether this statement is true, we need only to be able to parse “in \mathcal{K} ” for configurations in the language $\{P, Q, R_i\}_{i \leq 3+n}$. The conditions of being in \mathcal{K} is then described recursively in 0^{3+n-10} . \square

We use the following case of a theorem of Knight [12] to show that the countable model of T is recursively presentable.

Theorem 31. (*Knight*) *Let T be an \aleph_0 -categorical theory. If $T \cap \exists_{n+1}$ is Σ_n^0 uniformly in n , then T has a recursive model.*

Since T is \aleph_0 -categorical, Lemma 30 shows that T satisfies the conditions of this theorem. Thus we conclude the promised theorem.

Theorem 32. *Let \mathbf{d} be a tt -degree reducible to $0^{(\omega)}$. Then there exists an \aleph_0 -categorical theory $T \in \mathbf{d}$ in a finite language whose countable model is recursively presentable.*

Proof. As above, \mathbf{d} contains a set S such that $n \in S$ is uniformly recursive in $0^{(n)}$. Applying the construction to this set S yields an \aleph_0 -categorical theory $T \in \mathbf{d}$ in a finite language whose countable model is recursively presentable. \square

Corollary 33. *There exists an \aleph_0 -categorical theory T in a finite language whose countable model is recursively presentable and $T \equiv_T 0^{(\omega)}$.*

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