

COMPUTING AND DOMINATING THE RYLL-NARDZEWSKI FUNCTION

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ABSTRACT. We study, for a countably categorical theory T , the complexity of computing and the complexity of dominating the function specifying the number of n -types consistent with T .

1. INTRODUCTION

Independently in 1959, Erwin Engeler [3], Czesław Ryll-Nardzewski [10], and Lars Svenonius [12] provided a myriad of necessary and sufficient conditions on a first-order theory¹ for it to be countably categorical. Of these conditions, perhaps the best remembered is the existence of, for each $n \in \mathbb{N}$, only finitely many n -types consistent with the theory.

Definition. A theory T is *countably categorical* (alternately \aleph_0 -categorical) if T has, up to isomorphism, a unique countable model.

Ryll-Nardzewski Theorem (Engeler [3], Ryll-Nardzewski [10], and Svenonius [12]²). *A theory T is countably categorical if and only if there are only, for each $n \in \mathbb{N}$, finitely many n -types consistent with T .*

For a countably categorical theory T , the Ryll-Nardzewski Theorem implies the function mapping an integer n to the number of n -types consistent with T is a well-defined function from \mathbb{N} to \mathbb{N} . In this paper, we study the complexity of this function.

Definition. For an arbitrary theory T , the *Ryll-Nardzewski function* for T is the function $RN_T : \mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ such that $RN_T(n)$ gives the number of n -types consistent with T .

By the Ryll-Nardzewski Theorem, the function RN_T has range inside of \mathbb{N} if and only if T is countably categorical. The main result of this paper provides sharp upper bounds on the complexity of computing and the complexity of dominating RN_T for a countably categorical structure. Before stating this theorem, we recall the analogous result for a countably categorical theory.

Theorem 1 (Schmerl [11]). *Let T be a countably categorical theory. Then $RN_T \leq_T T'$. Moreover, this bound is sharp.*

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¹In this paper, a *theory* is always first-order, complete, and consistent.

²Though usually called the Ryll-Nardzewski Theorem, it should be noted that the result was independently and nearly simultaneously discovered by three mathematicians. It is only because of historical reasons that its name attributes it to Ryll-Nardzewski.

Proof. For any theory \mathbf{T} , we have $\text{RN}_{\mathbf{T}}(n) \geq m$ if and only if
 (†)

$$(\exists \psi_1(\bar{x})) \dots (\exists \psi_m(\bar{x})) \bigwedge_{1 \leq i \leq m} \mathbf{T} \vdash (\exists \bar{a}) [\psi_i(\bar{a})] \wedge (\forall \bar{x}) \left[\psi_i(\bar{x}) \implies \bigwedge_{j \neq i} \neg \psi_j(\bar{x}) \right],$$

with the $\psi_i(\bar{x})$ having exactly n -many free variables. For a countably categorical theory \mathbf{T} , the value of $\text{RN}_{\mathbf{T}}(n)$ is finite for all n by the Ryll-Nardzewski Theorem. Thus to compute $\text{RN}_{\mathbf{T}}(n)$, it suffices to find the greatest m such that $\text{RN}_{\mathbf{T}}(n) \geq m$. Since the outer conjunction in (†) is finitary, it is immediate that \mathbf{T}' suffices as an oracle to do so.

We refer the reader to the paper for sharpness. Alternately, it follows from Theorem 2. \square

Theorem 2. *There is a computable structure with countably categorical theory \mathbf{T} such that any function f dominating $\text{RN}_{\mathbf{T}}$ computes $\emptyset^{(\omega+1)}$. In particular, the Ryll-Nardzewski function $\text{RN}_{\mathbf{T}}$ satisfies $\text{RN}_{\mathbf{T}} \equiv_{\mathbf{T}} \emptyset^{(\omega+1)}$.*

By Theorem 1, this result is sharp. The proof of Theorem 2 is found in Section 2. Before delving into the proof of Theorem 2, we mention some related literature. Khossainov and Montalbán [6] construct a countably categorical theory \mathbf{T} such that $\mathbf{T} \equiv_{\mathbf{T}} \emptyset^{(\omega)}$. Andrews [1], for any $\mathbf{d} \leq_{tt} \emptyset^{(\omega)}$, constructs a countably categorical theory \mathbf{T} such that $\mathbf{T} \equiv_{tt} \mathbf{d}$ using a finite language. In both cases, however, there is a computable function f dominating $\text{RN}_{\mathbf{T}}$. Thus, those theories are inadequate to establish Theorem 2.

We refer the reader to Hodges [4] for background on model theory (especially Section 6.1 which covers Fraïssé Constructions) and to Ash and Knight [2] for background on computability theory and computable model theory.

2. PROOF OF THEOREM 2

Our construction of a theory \mathbf{T} witnessing Theorem 2 relies heavily on the existence of a $\mathbf{0}^{(\omega+1)}$ -computable function possessing an approximation satisfying various properties. In Section 2.1, we demonstrate the existence of such a function and approximation. In Section 2.2, we exhibit the model \mathcal{M} and verify it has the requisite properties.

2.1. The Function to Dominate. We include a proof of Lemma 3 as the form of h is important for showing Lemma 4.

Lemma 3 (Theorem 4.13 of Jockusch and McLaughlin [5]³). *There is a total $\emptyset^{(\omega+1)}$ -computable function $h : \mathbb{N} \rightarrow \mathbb{N}$ such that*

$$(\forall g : \mathbb{N} \rightarrow \mathbb{N}) \left[(\forall x \in \mathbb{N}) [g(x) > h(x)] \implies g \geq_{\mathbf{T}} \emptyset^{(\omega+1)} \right].$$

Proof. Let $h_1 : \mathbb{N} \rightarrow \mathbb{N}$ be the function given by

$$h_1(\langle i, j \rangle) := \begin{cases} s & \text{if } j \text{ enters } (\emptyset^{(i-1)})' \text{ at stage } s, \\ 0 & \text{otherwise, i.e., if } j \notin (\emptyset^{(i-1)})'. \end{cases}$$

³Though we reference Jockusch and McLaughlin [5] for the next result, it was known before then, at least implicitly. For example, it follows from the fact that for every $x \in \mathcal{O}$, there is a Π_1^0 -singleton f in Baire space with $f \equiv_{\mathbf{T}} H_x$ (Rogers [9]), and the fact that the Π_1^0 -singletons coincide with the uniformly majorreducible functions (Kuznetsov and Trahtenbrot [8]).

Let $h_2 : \mathbb{N} \rightarrow \mathbb{N}$ be the function given by

$$h_2(x) := (\mu s) \left[\emptyset^{(\omega+1)} \upharpoonright x = K^{\emptyset^{(\omega)}}[s] \upharpoonright x \right]$$

Define $h : \mathbb{N} \rightarrow \mathbb{N}$ by $h(x) := h_1(x) + h_2(x)$.

From its definition, it is immediate that $h \leq_T \emptyset^{(\omega+1)}$. Thus, we need only argue that any function g dominating h computes $\emptyset^{(\omega+1)}$. As a first step, we show that any function g dominating h_1 computes $\emptyset^{(\omega)}$. This is because, given i and j and using g , we can determine whether $j \in \emptyset^{(i)}$ by seeing if the computation $\varphi_j^{\emptyset^{(i-1)}}(j)[g(\langle i, j \rangle)]$ converges. Of course, the computation $\varphi_j^{\emptyset^{(i-1)}}(j)[g(\langle i, j \rangle)]$ converges if and only if $\varphi_j^{\emptyset^{(i-1)}}(j)[h_1(\langle i, j \rangle)]$ converges as g dominates h . The computation $\varphi_j^{\emptyset^{(i-1)}}(j)[g(\langle i, j \rangle)]$ may query $\emptyset^{(i-1)}$ as an oracle on a finite set of numbers. Having reduced the question whether j is in $\emptyset^{(i)}$ to a finite set of questions about $\emptyset^{(i-1)}$, repeating as such, we eventually reduce to questions about \emptyset , which are computable.

Thus, if g dominates h_1 , then it computes $\emptyset^{(\omega)}$. As a second step, we show that g computes $\emptyset^{(\omega+1)}$. This is because, given x , h_1 , and h_2 , we can determine whether $x \in \emptyset^{(\omega+1)}$ by computing $\varphi_x^{\emptyset^{(\omega)}}(x)[g(x)]$. As g dominates h_2 , this converges if and only if $x \in \emptyset^{(\omega+1)}$. Moreover, the computation is g -computable as g dominates h_1 and thus computes $\emptyset^{(\omega)}$. \square

When building the theory **T**, it will be necessary to approximate the function h . Though perhaps not strictly necessary, it simplifies later arguments if we impose strong constraints on how the approximations behave. Essentially, it is helpful to assume the approximations computed by $\emptyset^{(n)}$ for $n \in \mathbb{N}$ do not increase too quickly nor require the full computational power of the oracle.

Lemma 4. *There is a sequence of functions $\{f_n : \mathbb{N} \rightarrow \mathbb{N}\}_{n \in \mathbb{N}}$ such that:*

- (F1) *The function $f_n : \mathbb{N} \rightarrow \mathbb{N}$ is uniformly $\emptyset^{(n-4)}$ -computable.*
- (F2) *The functions $\{f_n\}_{n \in \mathbb{N}}$ satisfy $h(m) = \lim_{n \rightarrow \infty} f_n(m)$.*
- (F3) *The function f_0 satisfies $f_0(m) = 0$ for all m .*
- (F4) *The functions $\{f_n\}_{n \in \mathbb{N}}$ satisfy $f_n(n+3) = 0$.*
- (F5) *For all $n, m \in \mathbb{N}$, that $0 \leq f_{n+1}(m) - f_n(m) \leq 1$.*
- (F6) *For all $n \in \mathbb{N}$, that $|\{m : f_{n+1}(m) - f_n(m) = 1\}| \leq 1$.*

For notational convenience, we let $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be the function given by $f(m, n) := f_n(m)$.

Proof. It is enough to satisfy (F1) and (F2) since (F3), (F4), (F5), and (F6) can be easily achieved by slowing down and distributing any increases in the approximation. We describe how to approximate h_1 and h_2 separately, denoting their n th respective approximation function by $f_{1,n}$ and $f_{2,n}$. Then $f_n := f_{1,n} + f_{2,n}$ gives an approximation to h .

For approximating $h_1(\langle i, j \rangle)$, it suffices to take

$$f_{1,n}(\langle i, j \rangle) := \begin{cases} s & \text{if } n > i + 3 \text{ and } j \text{ enters } (\emptyset^{(i-1)})' \text{ at stage } s, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_{1,n}(\langle i, j \rangle)$ is uniformly $\emptyset^{(n-4)}$ -computable: The value is zero unless $n > i + 3$, in which case $n - 4 > i - 1$, so $\emptyset^{(n-4)}$ knows if and when j will enter $(\emptyset^{(i-1)})'$. Since $f_{1,n}(\langle i, j \rangle) = h_1(\langle i, j \rangle)$ if $n > i + 3$, we have $h_1(\langle i, j \rangle) = \lim_{n \rightarrow \infty} f_{1,n}(\langle i, j \rangle)$.

For approximating $h_2(x)$, it suffices to take

$$f_{2,n}(x) := (\mu s)(\forall j < x) \left[\text{if } j \in K^{\emptyset^{(\omega)}} \text{ with use contained in } \emptyset^{(n-5)}, \text{ then } j \in K_s^{\emptyset^{(\omega)}} \right].$$

Then $f_{2,n}(x)$ is uniformly $\emptyset^{(n-4)}$ -computable: For each j less than x , the oracle $\emptyset^{(n-4)}$ can determine if and when j enters $K^{\emptyset^{(\omega)}}$ with use contained in $\emptyset^{(n-5)}$. The value of $f_{2,n}(x)$ is then the maximum of the stages for those j that enter. Also, for any j , if j enters $K^{\emptyset^{(\omega)}}$, the computation uses at most a bounded number M_j of jumps. Letting M be the maximum of the number of such jumps for j less than x , i.e., letting $M := \max\{M_j\}_{j < x}$, we have $f_{2,n}(x) = h_2(x)$ for all $n > M + 1$. Thus $h_2(x) = \lim_{n \rightarrow \infty} f_{2,n}(x)$. \square

2.2. The Fraïssé Construction. In a manner similar to Andrews [1], we will employ a Fraïssé construction to create a countably categorical theory \mathbf{T} . The theory \mathbf{T} will be such that $\text{RN}_{\mathbf{T}}$ dominates the function h . The theory \mathbf{T} will be in a reduct of the language

$$L := \{U, V\} \cup \{R_i \mid i \in \omega, i \geq 3\} \cup \{Q_{j,k} \mid j, k \in \omega\},$$

where U and V are binary relations, R_i is an i -ary relation, and $Q_{j,k}$ is a j -ary relation.

The intuition is that the presence of the relation $Q_{j,k}$ (on some tuple) will code that $f(j, k) = f(j, k - 1) + 1$; the absence of the relation $Q_{j,k}$ (on every tuple) will code that $f(j, k) = f(j, k - 1)$. The remaining relations serve to create a countably categorical theory (after taking a Fraïssé Limit) such that the full theory is a definitional expansion of the theory restricted to the language $\{U, V, R_3\}$. Unfortunately, this intuition may be masked in the next definition to a reader unfamiliar with similar constructions.

Definition. Let \mathcal{K} be the class of finite L -structures \mathcal{C} where the following hold:

- (K1) Each relation on \mathcal{C} is symmetric and holds only on tuples of distinct elements.
- (K2) The structure \mathcal{C} satisfies

$$\neg(\exists \bar{x})(\exists y)(\exists z) \left[R_i(\bar{x}) \wedge U(y, z) \wedge \bigwedge_{\substack{\bar{w} \subset \bar{x} \\ |\bar{w}|=i-2}} (R_{i-1}(\bar{w}, y) \wedge R_{i-1}(\bar{w}, z)) \right].$$

- (K3) If $f(j, n) > f(j, n - 1)$, then \mathcal{C} satisfies

$$\neg(\exists x_1 \dots \exists x_j)(\exists y_1 \dots \exists y_{n-j}) \left[Q_{j,n}(\bar{x}) \wedge R_n(\bar{x}, \bar{y}) \wedge \bigwedge_{y_i, y_j} V(y_i, y_j) \right].$$

- (K4) If $f(j, n) = f(j, n - 1)$, then \mathcal{C} satisfies

$$\neg(\exists \bar{x}) [Q_{j,n}(\bar{x})].$$

To use the Fraïssé construction, we need to verify that \mathcal{K} has the hereditary property, the amalgamation property, and the joint embedding property.

Lemma 5. *The class \mathcal{K} satisfies the hereditary property, the amalgamation property, and the joint embedding property.*

Proof. The class \mathcal{K} has the hereditary property since it is defined via universal formulae.

For the amalgamation property, we show if \mathcal{A} , \mathcal{B} , and \mathcal{C} are L -structures in \mathcal{K} with $\mathcal{A} \subseteq \mathcal{B}, \mathcal{C}$, then there is an L -structure $\mathcal{D} \in \mathcal{K}$ and embeddings $g : \mathcal{B} \rightarrow \mathcal{D}$ and $h : \mathcal{C} \rightarrow \mathcal{D}$ with $g \upharpoonright_{\mathcal{A}} = h \upharpoonright_{\mathcal{A}}$. Fixing \mathcal{A} , \mathcal{B} , and \mathcal{C} , let \mathcal{D} be the free-join of \mathcal{B} and \mathcal{C} over \mathcal{A} , i.e., the structure with universe $B \cup C$ and with relations $R^{\mathcal{B} \cup \mathcal{C}} := R^{\mathcal{B}} \cup R^{\mathcal{C}}$ for any $R \in L$. Then \mathcal{D} satisfies (K1) as both \mathcal{B} and \mathcal{C} satisfy (K1). Also \mathcal{D} satisfies (K2), (K3), and (K4) as both \mathcal{B} and \mathcal{C} do and no relations hold in \mathcal{D} other than those in \mathcal{B} and \mathcal{C} . In particular, as every two elements in the disallowed tuple are in some realization of some relation, the disallowed tuple, were it to exist in \mathcal{D} , would have to be a subset of B or C . Thus we conclude $\mathcal{D} \in \mathcal{K}$, showing that \mathcal{K} has the amalgamation property.

Taking $A = \emptyset$, we see that \mathcal{K} has the joint embedding property. \square

Let \mathcal{M} be the unique Fraïssé limit (see, for example, Theorem 6.1.2 of [4]) of the class \mathcal{K} . The theory T we seek will be the theory of an appropriate reduct of \mathcal{M} . Since \mathcal{M} will be a definitional expansion of the reduct, we verify various facts about \mathcal{M} rather than the reduct.

Lemma 6. *The theory of \mathcal{M} is countably categorical. Hence, the theory of any reduct of \mathcal{M} is countably categorical.*

Proof. Being a Fraïssé limit, the structure \mathcal{M} is ultrahomogeneous, thus admits quantifier elimination. Thus, the number of n -types is determined by the number of quantifier-free n -types. As, for each n , there are only finitely many relations among P , R_i , and $Q_{j,k}$ which have arity at most n which have occurrences (since $h(n)$ is finite), the theory of \mathcal{M} is countably categorical.

Also, the reduct of any countably categorical theory is countably categorical. \square

Lemma 7. *The function $RN_{Th(\mathcal{M})}$ dominates h . Hence in any theory T for which $Th(\mathcal{M})$ is a definitional expansion of T , the function RN_T dominates h .*

Proof. Fixing j , by (F2), (F3), and (F5), there are at least $h(j)$ many n such that $f(j, n) > f(j, n-1)$. For each of these n , the relation $Q_{j,n}$ will hold on some tuple on which no other relation $Q_{j,n'}$ for $n' \neq n$ holds. Of course, this exploits the ultrahomogeneity of \mathcal{M} . Consequently, there are at least $h(j)$ many distinct n -types, so $RN_{Th(\mathcal{M})}(j) \geq h(j)$. \square

We now show that we can restrict our attention to an appropriate reduct of \mathcal{M} .

Lemma 8. *If $i > 3$, then*

$$\mathcal{M} \models (\forall \bar{x}) \left[R_i(\bar{x}) \iff \neg(\exists y)(\exists z) \left(U(y, z) \wedge \bigwedge_{\substack{\bar{w} \subseteq \bar{x} \\ |\bar{w}|=i-2}} (R_{i-1}(\bar{w}, y) \wedge R_{i-1}(\bar{w}, z)) \right) \right];$$

if $f(j, n) > f(j, n-1)$, then

$$\mathcal{M} \models (\forall \bar{x}) \left[Q_{j,n}(\bar{x}) \iff \neg(\exists \bar{y}) \left(R_n(\bar{x}, \bar{y}) \wedge \bigwedge_{y_i, y_j} V(y_i, y_j) \right) \right];$$

and if $f(j, n) = f(j, n-1)$, then

$$\mathcal{M} \models (\forall \bar{x}) [\neg Q_{j,n}(\bar{x})].$$

Thus, the structure \mathcal{M} is a definitional expansion of its reduct to the language $\{U, V, R_3\}$. Thus the reduct has the same Ryll-Nardzewski function as \mathcal{M} .

Proof. The rightward directions hold explicitly from (K2), (K3), and (K4). We show the leftward direction via the contrapositive.

Suppose $\mathcal{M} \models \neg R_i(\bar{x})$. By ultrahomogeneity, it suffices to see that there is some $\mathcal{C} \in \mathcal{K}$ which extends \bar{x} so that

$$\mathcal{C} \models (\exists y)(\exists z) \left[U(y, z) \wedge \bigwedge_{\substack{\bar{w} \subseteq \bar{x} \\ |\bar{w}|=i-2}} (R_{i-1}(\bar{w}, y) \wedge R_{i-1}(\bar{w}, z)) \right].$$

Let \mathcal{C} be the structure comprised of \bar{x} with two new elements y and z and whose relations are the relations on \bar{x} , the relation $U(y, z)$, and the relations $R_{i-1}(\bar{w}, y)$ and $R_{i-1}(\bar{w}, z)$ for \bar{w} a subset of \bar{x} of the appropriate size. As we added no occurrences of Q or V , we see that $\mathcal{C} \in \mathcal{K}$, and we are done.

Similarly, suppose $\mathcal{M} \models \neg Q_{j,n}(\bar{x})$. Let \mathcal{C} be the structure consisting of \bar{x} and a tuple \bar{y} whose relations are the relations on \bar{x} , $R_n(\bar{x}, \bar{y})$, and $V(y_i, y_j)$ for each $y_i, y_j \in \bar{y}$. It is easily seen that $\mathcal{C} \in \mathcal{K}$, and thus, by ultrahomogeneity of \mathcal{M} , that $\mathcal{M} \models (\exists \bar{y}) \left[R_n(\bar{x}, \bar{y}) \wedge \bigwedge_{y_i, y_j} V(y_i, y_j) \right]$. \square

We let \mathbf{T} be the theory of \mathcal{M} in the language with signature $\{U, V, R_3\}$. The reason for the reduct of \mathcal{M} is so that the countable model of \mathbf{T} is computable, which we will show using the following theorem.

Theorem 9 (Knight [7]). *Let \mathbf{T} be a countably categorical theory. If $\mathbf{T} \cap \exists_{n+1}$ is Σ_n^0 uniformly in n , then \mathbf{T} has a computable model.*

Lemma 10. *The reduct of the structure \mathcal{M} to the language $\{U, V, R_3\}$ is computable.*

Proof. Uniformly in n , the fragment $\mathbf{T} \cap \exists_n$ is computable in $\emptyset^{(n-1)}$. The salient point is that n -quantifier formulae in \mathbf{T} are equivalent to quantifier-free formulae in the language $\{U, V\} \cup \{R_i \mid i \leq n+3\} \cup \{Q_{j,k} \mid k \leq n+2\}$. The n -quantifier theory of \mathbf{T} is thus determined by whether or not $f(j, k) > f(j, k-1)$ for $k \leq n+2$. This in turn uniformly depends on information computable in $\emptyset^{(n+2-4)} = \emptyset^{(n-2)}$.

It remains to see that the n -quantifier formulae in \mathbf{T} are equivalent to quantifier-free formulae in the language $\{U, V\} \cup \{R_i \mid i \leq n+3\} \cup \{Q_{j,k} \mid k \leq n+2\}$. This follows by playing an Ehrenfeucht-Fraïssé game of length n . Given a pair of tuples \bar{a}, \bar{b} which have the same quantifier-free $\{U, V\} \cup \{R_i \mid i \leq n+3\} \cup \{Q_{j,k} \mid k \leq n+2\}$ -types, and given a tuple \bar{c} , it suffices to show the existence of a tuple \bar{d} so that $\bar{a}\bar{c}$ and $\bar{b}\bar{d}$ have the same $\{U, V\} \cup \{R_i \mid i \leq n+2\} \cup \{Q_{j,k} \mid k \leq n+1\}$ -types. It is easy to check that such a $\bar{b}\bar{d}$ exists in \mathcal{K} , and the rest is done by ultrahomogeneity of \mathcal{M} . \square

Taken together, Lemma 3, Lemma 6, Lemma 7, and Lemma 10 show the theory \mathbf{T} witnesses Theorem 2.

REFERENCES

- [1] Uri Andrews. The degrees of categorical theories with recursive models. *Proc. Amer. Math. Society*, to appear.

- [2] C. J. Ash and J. Knight. *Computable structures and the hyperarithmetical hierarchy*, volume 144 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 2000.
- [3] Erwin Engeler. Äquivalenzklassen von n -Tupeln. *Z. Math. Logik Grundlagen Math.*, 5:340–345, 1959.
- [4] Wilfrid Hodges. *A shorter model theory*. Cambridge University Press, Cambridge, 1997.
- [5] C. G. Jockusch, Jr. and T. G. McLaughlin. Countable retracing functions and Π_2^0 predicates. *Pacific J. Math.*, 30:67–93, 1969.
- [6] Bakhadyr Khossainov and Antonio Montalbán. A computable \aleph_0 -categorical structure whose theory computes true arithmetic. *J. Symbolic Logic*, 75(2):728–740, 2010.
- [7] Julia F. Knight. Nonarithmetical \aleph_0 -categorical theories with recursive models. *J. Symbolic Logic*, 59(1):106–112, 1994.
- [8] A. V. Kuznecov and B. A. Trahtenbrot. Investigation of partially recursive operators by means of the theory of Baire space. *Dokl. Akad. Nauk SSSR (N.S.)*, 105:897–900, 1955.
- [9] Laurel A. Rogers. Ulm’s theorem for partially ordered structures related to simply presented abelian p -groups. *Trans. Amer. Math. Soc.*, 227:333–343, 1977.
- [10] C. Ryll-Nardzewski. On the categoricity in power $\leq \aleph_0$. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys.*, 7:545–548. (unbound insert), 1959.
- [11] James H. Schmerl. A decidable \aleph_0 -categorical theory with a nonrecursive Ryll-Nardzewski function. *Fund. Math.*, 98(2):121–125, 1978.
- [12] Lars Svenonius. \aleph_0 -categoricity in first-order predicate calculus. *Theoria (Lund)*, 25:82–94, 1959.

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