

SPECTRA OF COMPUTABLE MODELS OF STRONGLY MINIMAL DISINTEGRATED THEORIES IN LANGUAGES OF BOUNDED ARITY

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ABSTRACT. In this paper, we consider the spectra of computable (or recursive) models of strongly minimal disintegrated theories. We conjecture that for any m , there are only finitely many spectra of computable models of strongly minimal disintegrated \mathcal{L} -theories T in computable (but possibly infinite) relational languages \mathcal{L} of arity at most m . We verify this conjecture here for binary and ternary languages and present some further partial results toward proving the general conjecture. We also determine the exactly seven spectra possible for binary languages and show that for ternary languages, there are at least nine but no more than eighteen spectra.

1. INTRODUCTION

Classifying the computable (or recursive) models of a given first-order theory (over a computable language) has been a long-standing problem in computable model theory. The problem can be stated particularly succinctly for uncountably but not totally categorical theories T , since in that case, the countable models of T form an elementary chain $\mathcal{M}_0 \prec \mathcal{M}_1 \prec \cdots \prec \mathcal{M}_\omega$, where \mathcal{M}_α is the model of dimension $k + \alpha$ and where k is the dimension of the prime model. Classifying the computable models of such T can now be phrased in terms of subsets of $[0, \omega] = \omega + 1 = \omega \cup \{\omega\}$; namely, we call the set $\text{SRM}(T) = \{\alpha \leq \omega \mid \mathcal{M}_\alpha \text{ is computable}\}$ the *spectrum of computable models* of T .

This spectrum problem has been a motivating question in the field since Goncharov [Go78] proved that there are non-trivial spectra. Since then, only incremental progress has been made in classifying the spectra of uncountably but not totally categorical theories. In fact, very few negative results are known; and no results are known separating uncountably categorical theories from the more restrictive notion of strongly minimal theories; so in this paper, we will restrict our attention to the latter. (We will also tacitly assume from now on that all languages are computable.)

What is known is now contained in the following theorems:

Theorem 1. *The following are spectra of computable model of strongly minimal theories:*

- \emptyset

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- $[0, \omega]$, i.e., $\omega + 1$
- $\{0\}$ (Goncharov [Go78])
- $[0, n]$, i.e., $\{0, \dots, n\}$ for any $n \in \omega$ (Kudaibergenov [Ku80])
- $[0, \omega]$, i.e., ω (Khoussainov/Nies/Shore [KNS97])
- $[1, \omega]$, i.e., $(\omega + 1) \setminus \{0\}$ (Khoussainov/Nies/Shore [KNS97])
- $\{1\}$ (Nies [Ni99])
- $[1, \alpha)$ for any $\alpha \in [2, \omega]$ (Nies/Hirschfeldt, see Nies [Ni99, p. 314])
- $\{\omega\}$ (Hirschfeldt/Khoussainov/Semukhin [HKS06])
- $\{0, \omega\}$ (Andrews [An11])

Despite the relatively few sets known to be spectra, there are also very few general upper bounds on the complexity of spectra.

Theorem 2. *Let T be a strongly minimal theory.*

- (Nies [Ni99]) *The spectrum of computable models of an uncountably categorical theory T is $\Sigma_{\omega+3}^0$, i.e., it is $\Sigma_3^0(\emptyset^{[\omega]})$.*
- (Nies [Ni99]) *The spectrum of computable models of a model complete uncountably categorical theory is Σ_4^0 .*
- (Goncharov/Harizanov/Laskowski/Lempp/McCoy [GHLLM03]) *The spectrum of computable models of a strongly minimal disintegrated theory T is Σ_5^0 .*

Even in the geometrically simplest case of a strongly minimal theory, namely, that of a binary language, the gap remains very large:

Theorem 3. *The following are spectra of computable models of strongly minimal theories in binary languages:*

- \emptyset
- $[0, \omega]$
- $\{0\}$ (Goncharov [Go78])
- $\{0, 1\}$ (Kudaibergenov [Ku80])
- $[1, \omega]$ (Khoussainov/Nies/Shore [KNS97])
- $\{1\}$ (Nies [Ni99])
- $\{\omega\}$ (Hirschfeldt/Khoussainov/Semukhin [HKS06])

The only known upper bound is the above result that the spectrum must be Σ_5^0 (since for binary languages, a strongly minimal theory must be disintegrated).

Further specializing from strongly minimal to strongly minimal disintegrated theories, a much stronger result is possible in the case of finite languages:

Theorem 4 (Andrews/Medvedev [AM14]). *The only possible spectra of computable models of a strongly minimal disintegrated theory in a finite language are \emptyset , $[0, \omega]$ and $\{0\}$. (They also show the same result for modular groups and note that it follows from Poizat [Po88] that the only spectra for field-like fields in finite languages are \emptyset and $[0, \omega]$.)*

The positive direction, showing that $\{0\}$ is a possible spectrum of a strongly minimal theory in a finite (binary) language, was obtained by Herwig, Lempp and Ziegler [HLZ99].

The next step is to allow the language to be infinite but to restrict the arity of the (relational) language, since all known examples seem to suggest that one needs to increase the arity more and more to obtain more spectra.

In section 2, we will first show that in the case of binary languages (in which case the theory must be disintegrated), the seven spectra in Theorem 3 are the only possible spectra for strongly minimal (and thus disintegrated) theories.

In section 3, we will extend this to show that for strongly minimal disintegrated theories in ternary languages, there are at least nine but no more than eighteen possible spectra. We also present some partial results for higher bounded arities.

These results lead us to the following sweeping

Conjecture 5. *For $m \geq 1$, there are only finitely many spectra of computable models of strongly minimal disintegrated theories T in computable relational languages of arity at most m .*

2. BINARY LANGUAGES

The purpose of this section is to prove the following

Theorem 6. *The following are exactly the seven possible spectra of computable models of strongly minimal disintegrated theories in binary languages: \emptyset , $[0, \omega]$, $\{0\}$, $\{1\}$, $\{0, 1\}$, $[1, \omega]$, and $\{\omega\}$.*

Proof. First of all, Theorem 3 shows that the seven spectra above are indeed realized in binary languages.

We now first reduce the number of possible spectra to a small finite list and then analyze cases. Throughout this section, we will assume an infinite computable binary relational language \mathcal{L} which (in any \mathcal{L} -theory T considered) is effectively closed under permutation of variables, i.e., that from an index of $R(x, y) \in \mathcal{L}$, we can effectively find an index for $R(y, x)$.

Lemma 7. *For a strongly minimal \mathcal{L} -theory T , if $k \in \text{SRM}(T)$ for some $k \in [2, \omega]$, then the set of relations in \mathcal{L} which are of rank 2 is a computable set. Thus T is computably interdefinable with a theory all relation symbols of which have rank at most 1.*

Proof. Let \mathcal{N} be a computable presentation of \mathcal{M}_k for some $k \geq 2$, and let $a, b \in N$ be a generic pair in N . Then a relation R has rank 2 if and only if $\mathcal{N} \models R(a, b)$. So the rank-2 relations R form a computable set, and each such R can be replaced by $\neg R$, which has rank at most 1. \square

Thus, in the case where $\text{SRM}(T)$ contains an element ≥ 2 , we will assume from now on that every relation symbol has rank ≤ 1 .

For the remainder of the proof, we need the following

Definition 8. *Fix a model \mathcal{M} in a binary relational language $\mathcal{L} = \{R_0, R_1, \dots\}$ such that the Morley rank of each relation is at most 1.*

- (1) *The 0-neighborhood of an element $a \in M$ is $\text{Nbh}_0(a) = \{a\}$.*
- (2) *The $(n + 1)$ -neighborhood $\text{Nbh}_{n+1}(a)$ of $a \in M$ is the union of the n -neighborhood $\text{Nbh}_n(a)$ and the set of all elements $d \in M$ such that there are $c \in \text{Nbh}_n(a)$ and a relation R_i (with $i \leq n$) with*

$$\begin{aligned} & [R_i(c, d) \wedge \neg \exists^\infty y R_i(c, y) \wedge \neg \exists^\infty x R_i(x, d)] \vee \\ & [\neg R_i(c, d) \wedge (\exists^\infty y R_i(c, y) \vee \exists^\infty x R_i(x, d))]. \end{aligned}$$

- (3) *The neighborhood $\text{Nbh}(a)$ of an element $a \in M$ is the union of all n -neighborhoods of a .*

(Recall that we assume the language \mathcal{L} to be closed under permutation of variables.)

Before we embark on the rest of our proof, recall the structure of a strongly minimal model \mathcal{M} in our language \mathcal{L} : Fixing a basis B of \mathcal{M} , the model will be the disjoint union of the algebraic closure $\text{acl}(\emptyset)$ and the interalgebraic closures $\text{iacl}(a) = \text{Nbh}(a)$ for each $a \in B$ (by Herwig/Lempp/Ziegler [HLZ99] and Andrews/Medvedev [AM14]). Note that all sets $\text{iacl}(a)$ will be isomorphic, and 1-transitive in \mathcal{M} ; they may be finite or countably infinite. The algebraic closure $\text{acl}(\emptyset)$ may be finite or countably infinite; it may even be empty. If $\text{acl}(\emptyset)$ is finite, then the interalgebraic closures $\text{iacl}(a)$ of all generic elements a must be infinite since T is not totally categorical, and \mathcal{M}_k has a basis of size $1+k$; otherwise \mathcal{M}_k has a basis of size k .

The following observation will be used repeatedly throughout this section.

Observation 9. *If \mathcal{M}_k is a computable model of T for $k \geq 2$ and a is any element of \mathcal{M} , then $\text{Nbh}(a)$ is a Σ_1^0 -set.*

Proof. The definition of $\text{Nbh}(a)$ is Σ_1^0 over the set of (R, x) so that $\exists^\infty y R(x, y)$ (using the effective closure of \mathcal{L} under permutation of variables). To see that this set is computable, fix a pair of mutually generic elements a, b and note that $\mathcal{M} \models \exists^\infty y R(x, y)$ if and only if $\mathcal{M} \models R(x, a) \wedge R(x, b)$. \square

We now isolate a key step of Lemma 11 below, as it will reappear below.

Lemma 10. *Let \mathcal{N} be a computable strongly minimal structure (in any computable relational language \mathcal{L}). Let M be a Δ_2^0 -subset of N , and let $A \subseteq M$ be any infinite Σ_1^0 -set. Then there is a computable copy of \mathcal{M} .*

Proof. We may assume that $N \setminus M$ is infinite, else the result is trivial. Let M_s be the set of elements believed to be in M at stage s . We construct \mathcal{M} by copying elements and relations from \mathcal{N} . At stage s , we will have copied a finite set $X \subseteq N$ and the quantifier-free diagram of X in a finite sublanguage \mathcal{L}_0 . At this stage s , we then want to reassign which elements we are copying so as to only be copying elements in M_s . Let $Y = X \setminus M_s$. Either the approximation to M_s changes on some elements of X , or else for any $y_0 \in Y$,

$$\phi(z) := \exists Z ((Z, X \setminus Y, z) \cong_{\mathcal{L}_0} (Y, X \setminus Y, y_0))$$

is an existential first-order formula in the type of y_0 over $X \setminus Y$. But if y_0 is not in M_s , then, by strong minimality, all but finitely many elements in N satisfy $\phi(z)$. Thus we eventually see an element $a \in A$ which satisfies ϕ . We reassign y_0 to be copying a . Repeating this process, eventually Y is empty.

At every stage s , we have built a finite structure X in a finite language \mathcal{L}_0 and have an embedding $f_s : X \rightarrow M_s$. If $x \in M$, then from some point onward $x \in \text{ran}(f_s)$. If $x \notin M$, then from some point onwards $x \notin \text{ran}(f_s)$. Thus, in the limit, we construct a copy of \mathcal{M} . \square

Our next lemma reduces the number of possible spectra to a small finite number of possibilities:

Lemma 11. *For a strongly minimal \mathcal{L} -theory T , if $k \in \text{SRM}(T)$ for $2 \leq k < \omega$, then $[1, \omega] \subseteq \text{SRM}(T)$.*

Proof. Fix a computable presentation \mathcal{N} of the model \mathcal{M}_k . We first show that $k+1 \in \text{SRM}(T)$. Fix a generic pair of elements a, b from \mathcal{N} . Then $\text{iacl}(a) = \text{Nbh}(a)$ is a Σ_1^0 -set; so there is a computable structure \mathcal{C} isomorphic to $\text{Nbh}(a)$. We create a computable presentation \mathcal{M} of \mathcal{M}_{k+1} , which is the disjoint union of a copy of \mathcal{C} along with a copy of \mathcal{N} . We declare a relation R to hold on the pair (c, d) for $c \in \mathcal{C}$ and $d \in \mathcal{N}$ iff $\mathcal{N} \models R(a, d) \wedge R(b, d)$ (which extends to a full definition of all relations by our convention that \mathcal{L} is effectively closed under permutation of variables). It is easy to see that \mathcal{M} is isomorphic to \mathcal{M}_{k+1} , thus $k+1 \in \text{SRM}(T)$. As the procedure to take a presentation of \mathcal{M}_k and produce a presentation of \mathcal{M}_{k+1} is uniformly computable, we see that $\omega \in \text{SRM}(T)$ as well.

It thus remains to verify that $\text{SRM}(T)$ contains $k-1$. The proof distinguishes two cases:

Case 1: $\text{Nbh}(a)$ is finite: In this case, there is a computable (even cofinite) subset of \mathcal{N} isomorphic to \mathcal{M}_{k-1} .

Case 2: $\text{Nbh}(a)$ is infinite: In this case, \mathcal{M}_{k-1} is a Δ_2^0 - (in fact, a Π_1^0 -)subset of \mathcal{M}_k . Furthermore, it contains an infinite Σ_1^0 -set $\text{Nbh}(a)$. Thus, by Lemma 10, \mathcal{M}_{k-1} has a computable copy. \square

Corollary 12. *Every possible spectrum of a binary strongly minimal theory is among the following ten: \emptyset , $[0, \omega]$, $[1, \omega]$, $\{\omega\}$, $\{0\}$, $\{1\}$, $\{0, 1\}$, $\{0, \omega\}$, $\{1, \omega\}$, and $\{0, 1, \omega\}$.*

The first seven are known to be spectra of binary strongly minimal theories by Theorem 3. We will now show that the last three, $\{0, \omega\}$, $\{1, \omega\}$ and $\{0, 1, \omega\}$, are not spectra of binary strongly minimal theories:

Lemma 13. *If $\omega \in \text{SRM}(T)$, and one of $0 \in \text{SRM}(T)$ or $1 \in \text{SRM}(T)$, then $2 \in \text{SRM}(T)$, and so $[1, \omega] \subseteq \text{SRM}(T)$ by Lemma 11.*

Proof. We fix a computable presentation \mathcal{N} of the model \mathcal{M}_ω and a computable presentation \mathcal{M} of the model \mathcal{M}_0 or \mathcal{M}_1 . We distinguish two cases.

Case 1: There is no element $c \in \text{acl}_{\mathcal{M}}(\emptyset)$ so that for each relation R , both $\neg \exists^\infty y R(y, c)$ and $\forall x (\exists^\infty y R(x, y) \rightarrow R(x, c))$ hold. (Recall again that we assume \mathcal{L} to be closed under permutation of variables.)

In this case, we fix generic elements a and b in \mathcal{N} . We then have

$$\text{acl}_{\mathcal{N}}(\emptyset) = \{d \mid \exists R \in \mathcal{L} [(R(a, d) \wedge R(b, d)) \vee \exists x (R(x, a) \wedge R(x, b) \wedge \neg R(x, d))]\}.$$

Thus $\text{acl}_{\mathcal{N}}(\emptyset)$ is a Σ_1^0 -subset of N . We have already seen that the inter-algebraic closure of a generic singleton in \mathcal{N} is Σ_1^0 , thus a computable presentation of \mathcal{M}_2 can be given as $\text{acl}_{\mathcal{N}}(\emptyset) \cup \text{iacl}_{\mathcal{N}}(a) \cup \text{iacl}_{\mathcal{N}}(b)$, which is a Σ_1^0 -subset of \mathcal{N} and thus a computably presentable model.

Case 2: Otherwise, there is an element $c \in \text{acl}_{\mathcal{M}}(\emptyset)$ so that for each relation R , both $\neg \exists^\infty y R(y, c)$ and $(\exists^\infty y R(x, y) \rightarrow R(x, c))$ hold. We fix such an element c and further distinguish two subcases.

Subcase 2.1: There is an element $d \in M$ so that for each relation R ,

$$\mathcal{M} \models \forall x (\exists^\infty y R(x, y) \leftrightarrow (R(x, c) \wedge R(x, d))).$$

(This will in particular be the case if \mathcal{M} is a presentation of \mathcal{M}_1 since we can choose a generic element d .) We now expand \mathcal{M} by defining new computable unary predicates A_i by letting $A_i(x)$ hold for $x \in M$ iff $\mathcal{M} \models R_i(x, c) \wedge R_i(x, d)$. Then by the assumptions of Subcase 2.1, $A_i(x)$ holds iff $\mathcal{M} \models \exists^\infty y R(x, y)$. As in Case 1, we

can fix computable presentations of $\text{iacl}(a)$ and $\text{iacl}(b)$. We now define a computable presentation of \mathcal{M}_2 as the disjoint union of \mathcal{M} and $\text{iacl}(a)$ (as well as $\text{iacl}(b)$ if \mathcal{M} is a computable presentation of \mathcal{M}_0), letting $R_i(x, y)$ hold for $x \in M$ and $y \in \text{iacl}(a)$ or $y \in \text{iacl}(a) \cup \text{iacl}(b)$, respectively, iff $A_i(x)$ holds, and letting $R_i(x, y)$ never hold for $x \in \text{iacl}(a)$ and $y \in \text{iacl}(b)$.

Subcase 2.2: Otherwise,

$$M = \{e \mid \exists R \in \mathcal{L} \exists x (\mathcal{M} \models \exists^\infty y R(x, y) \not\leftrightarrow (R(x, c) \wedge R(x, e)))\}$$

(since, as noted in Subcase 2.1, we must have $\mathcal{M} = \mathcal{M}_0$ in Subcase 2.2). Let $c' \in N$ be the image of c under an elementary embedding of \mathcal{M} into \mathcal{N} , and again fix a pair of generic elements a and b in \mathcal{N} . Then

$$\text{acl}_{\mathcal{N}}(\emptyset) = \{e \mid \exists R \in \mathcal{L} \mathcal{N} \models \exists x ((R(x, a) \wedge R(x, b)) \not\leftrightarrow (R(x, c') \wedge R(x, e)))\},$$

which is a Σ_1^0 -set. But then, as in Case 1, the disjoint union of $\text{acl}_{\mathcal{N}}(\emptyset)$, $\text{Nbh}(a)$ and $\text{Nbh}(b)$ is a Σ_1^0 -subset of N isomorphic to \mathcal{M}_2 . \square

This concludes the proof of Theorem 6. \square

3. TERNARY LANGUAGES AND LANGUAGES OF HIGHER ARITY

The purpose of this section is to show in Theorem 14 that, as for spectra of computable models of strongly minimal disintegrated theories in binary relations, we have only finitely many possible spectra of computable models in the case of ternary relations as well. On the other hand, Kudaibergenov [Ku80] showed that $\{0, 1, 2\}$ is the spectrum of a strongly minimal disintegrated theory in a ternary language, and Nies/Hirschfeldt [Ni99, p.314] showed that $\{1, 2\}$ is the spectrum of a strongly minimal disintegrated theory in a ternary language, neither of which, by our Theorem 6, is possible in the binary case. This leads to our Corollary 15, giving the current best upper and lower bounds on the number of spectra of computable models of strongly minimal disintegrated theories in ternary relations.

Theorem 14. *If T is a strongly minimal disintegrated theory in a language of ternary relations and $\text{SRM}(T) \cap [3, \omega] \neq \emptyset$, then $[1, \omega] \subseteq \text{SRM}(T)$. Thus there are at most eighteen spectra of computable models of disintegrated strongly minimal theories in ternary languages.*

Theorems 6 and 14 as well as the above-mentioned results of Kudaibergenov and of Nies/Hirschfeldt allow us to immediately conclude the following

Corollary 15. *There are at least nine but at most eighteen subsets of $[0, \omega]$ which are spectra of computable models of a strongly minimal disintegrated theory T in a language of ternary relations.* \square

Proof of Theorem 14. The proof proceeds in a number of lemmas. The first lemma is the analog of Lemma 7, and the proof is the same.

Lemma 16. *For a strongly minimal \mathcal{L} -theory T , if $k \in \text{SRM}(T)$ and $k \in [3, \omega]$, then the set of relations in \mathcal{L} which are of rank 3 is a computable set. Thus T is computably interdefinable with a theory all relation symbols of which have rank at most 2.* \square

Using this lemma, whenever we have $\text{SRM}(T) \cap [3, \omega] \neq \emptyset$, we will assume that all relation symbols in \mathcal{L} have Morley rank ≤ 2 .

We now use a new trick, which does not seem to generalize to arity > 3 , to get down to Morley rank at most 1. Note that a similar reduction to rank 1 for arbitrary theories of bounded arity would, by Theorem 24 below, suffice to prove Conjecture 5.

Lemma 17. *For each ternary relation R of Morley rank ≤ 2 in a strongly minimal disintegrated theory, the three relations $\exists^\infty w R(w, y, z)$, $\exists^\infty w R(x, w, z)$, and $\exists^\infty w R(x, y, w)$ all have Morley rank ≤ 1 , as does the symmetric difference between $R(x, y, z)$ and $\exists^\infty w R(w, y, z) \vee \exists^\infty w R(x, w, z) \vee \exists^\infty w R(x, y, w)$.*

Proof. If y, z were to be mutually generic where $\exists^\infty w R(w, y, z)$, then for any x generic over y, z , we would have $R(x, y, z)$, but then R would have Morley rank 3. Thus $\exists^\infty w R(w, y, z)$ has Morley rank ≤ 1 . A symmetric argument works for $\exists^\infty w R(x, w, z)$ and $\exists^\infty w R(x, y, w)$.

Suppose

$$R(x, y, z) \wedge \neg[\exists^\infty w R(w, y, z) \vee \exists^\infty w R(x, w, z) \vee \exists^\infty w R(x, y, w)]$$

has Morley rank 2. Then, by symmetry, there would be two mutually generic elements x and y as well as an element z so that $R(x, y, z)$ and $\neg\exists^\infty w R(x, y, w)$. Thus $z \in \text{acl}(x, y)$. Using the fact that our theory is disintegrated, we may assume without loss of generality that $z \in \text{acl}(y)$. Then x is generic over y, z , and so $\exists^\infty w R(w, y, z)$, which is a contradiction.

Suppose

$$[\exists^\infty w R(w, y, z) \vee \exists^\infty w R(x, w, z) \vee \exists^\infty w R(x, y, w)] \wedge \neg R(x, y, z)$$

has Morley rank 2. Then, without loss of generality, for some mutually generic x and y and some element z ,

$$[\exists^\infty w R(w, y, z) \vee \exists^\infty w R(x, w, z) \vee \exists^\infty w R(x, y, w)] \wedge \neg R(x, y, z)$$

holds. The third disjunct is impossible, as then R would be rank 3. So, without loss of generality, $\exists^\infty w R(w, y, z) \wedge \neg R(x, y, z)$. Thus $x \in \text{acl}(y, z)$. Since x and y are mutually generic, $x \in \text{acl}(z)$. Thus z is generic over y . But then $\exists^\infty w R(w, y, z)$ implies that R has rank 3, again a contradiction. \square

Given a language \mathcal{L} comprised of ternary relations of rank ≤ 2 (which we may assume if $[3, \omega] \cap \text{SRM}(T) \neq \emptyset$), we define

$$\begin{aligned} \mathcal{L}' = & \{\exists^\infty w R(w, y, z) \mid R \in \mathcal{L}\} \cup \\ & \{R(x, y, z) \wedge \neg[\exists^\infty w R(w, y, z) \vee \exists^\infty w R(x, w, z) \vee \exists^\infty w R(x, y, w)] \mid R \in \mathcal{L}\} \cup \\ & \{[\exists^\infty w R(w, y, z) \vee \exists^\infty w R(x, w, z) \vee \exists^\infty w R(x, y, w)] \wedge \neg R(x, y, z) \mid R \in \mathcal{L}\}. \end{aligned}$$

Then \mathcal{L}' is a language inter-definable with \mathcal{L} ; in fact, every relation in \mathcal{L} is even a Boolean combination of relations in \mathcal{L}' , and all relation symbols in \mathcal{L}' have Morley rank ≤ 1 . Moreover, we have the following

Lemma 18. *If \mathcal{M}_k is a computable model for $k \geq 3$ of a strongly minimal disintegrated theory T , then each relation in \mathcal{L}' is uniformly computable in \mathcal{M}_k .*

Proof. Fix mutually generic elements a, b, c in \mathcal{M}_k . Without loss of generality, we only need to computably determine whether a pair a', b' satisfies $\exists^\infty w R(a', b', w)$. Note that at least one of a, b, c is generic over a', b' .

Case 0: For each $e \in \{a, b, c\}$, $\neg R(a', b', e)$. Then $\neg \exists^\infty w R(a', b', w)$ since an element which is generic over a', b' satisfies $\neg R(a', b', e)$.

Case 1: For exactly one $e \in \{a, b, c\}$, $R(a', b', e)$ holds. If a' and b' are mutually generic, then $\neg \exists^\infty w R(a', b', w)$ since R has rank ≤ 2 . If a' and b' are not mutually generic, then at least two elements in $\{a, b, c\}$ are generic over a', b' . Thus these two elements either both realize $R(a', b', z)$, or neither does. Since exactly one element of $\{a, b, c\}$ realizes $R(a', b', z)$, neither of these two elements which is generic over a', b' can realize $R(a', b', z)$. Thus again $\neg \exists^\infty w R(a', b', w)$. So $\neg \exists^\infty w R(a', b', w)$ holds independently of whether a' and b' are mutually generic or not.

Case 2: For exactly two $e \in \{a, b, c\}$, $R(a', b', e)$ holds; say,

$$R(a', b', a) \wedge R(a', b', b) \wedge \neg R(a', b', c).$$

In this case, we need to further distinguish subcases in order to determine whether $\exists^\infty w R(a', b', w)$ holds or not.

Subcase 2.1: If $\neg \exists^\infty w R(a', b', w)$, then both a and b are algebraic over a', b' ; and so a', b' must be mutually generic as well. So, without loss of generality, by symmetry, a' is interalgebraic with a and b' is interalgebraic with b . Thus $R(a', b', a)$ implies that $R(a', b, a)$, and $R(a', b', b)$ implies that $R(a, b', b)$. So, we have $R(a', b, a) \wedge R(a, b', b)$. (Symmetrically, if a' is interalgebraic with b and b' is interalgebraic with a , then we have $R(a', a, b) \wedge R(b, b', a)$.)

Subcase 2.2: If $\exists^\infty w R(a', b', w)$, then a', b' must have rank at most 1 since the rank of R is at most 2; and c is algebraic over a', b' , so by exchange, $a', b' \in \text{acl}(c)$, whereas a and b are both generic over a', b' . Thus if $R(a', b, a)$ holds, then $a' \in \text{acl}(a, b) \cap \text{acl}(c) = \text{acl}(\emptyset)$. Similarly, if $R(a, b', b)$ holds, then $b' \in \text{acl}(\emptyset)$. These cannot both happen since then $c \in \text{acl}(a', b') = \text{acl}(\emptyset)$. Symmetrically, $R(a', a, b)$ and $R(b, b', a)$ cannot both happen.

In conclusion, we have that $\neg \exists^\infty w R(a', b', w)$ holds iff

$$(R(a', b, a) \wedge R(a, b', b)) \vee (R(a', a, b) \wedge R(b, b', a)).$$

Case 3: For each $e \in \{a, b, c\}$, $R(a', b', e)$ holds. Then since one of them is generic over a', b' , $\exists^\infty w R(a', b', w)$ holds.

Thus, in each case, we can computably in the atomic diagram of \mathcal{M}_k determine whether or not $\exists^\infty w R(a', b', w)$ holds for any pair of elements a', b' . \square

We next need to adjust Definition 8 of a neighborhood of an element to higher arity:

Definition 19. Fix a model \mathcal{M} in a relational language $\mathcal{L} = \{R_0, R_1, \dots\}$ such that the Morley rank of each relation is at most 1. For each relation symbol $R \in \mathcal{L}$ of arity m , say, define $R^{j,l}$ (for $1 \leq j < l \leq m$) as the projection of R onto its j th and l th coordinates, i.e., $R^{j,l}(x_j, x_l)$ holds iff

$$\exists x_1 \dots \exists x_{j-1} \exists x_{j+1} \dots \exists x_{l-1} \exists x_{l+1} \dots \exists x_m R(\bar{x}).$$

We now define the neighborhood of $a \in M$ by recursion as follows:

- (1) The 0-neighborhood of an element $a \in M$ is $\text{Nbh}_0(a) = \{a\}$.
- (2) The $(n+1)$ -neighborhood $\text{Nbh}_{n+1}(a)$ of $a \in M$ is the union of the n -neighborhood $\text{Nbh}_n(a)$ and the set of all elements $d \in M$ such that there are $c \in \text{Nbh}_n(a)$, a relation R_i (with $i \leq n+1$) of arity m , say, and

$1 \leq j < l \leq m$ with

$$\begin{aligned} & [R_i^{j,l}(c, d) \wedge \neg \exists^\infty y R_i^{j,l}(c, y) \wedge \neg \exists^\infty x R_i^{j,l}(x, d)] \vee \\ & [\neg R_i^{j,l}(c, d) \wedge (\exists^\infty y R_i^{j,l}(c, y) \vee \exists^\infty x R_i^{j,l}(x, d))]. \end{aligned} \quad (*)$$

(3) The neighborhood $\text{Nbh}(a)$ of an element $a \in M$ is the union of all n -neighborhoods of a .

(Recall that we assume the language \mathcal{L} to be closed under permutation of variables.)

We now first state a useful technical

Lemma 20. *Fix a model \mathcal{M}_k of T for $k \in [3, \omega]$ with a computable presentation in a relational language $\mathcal{L} = \{R_0, R_1, \dots\}$ such that the Morley rank of each relation is at most 1. Then \mathbf{O}' can compute the canonical index of the finite set of elements $d \in M_k$ so that $\exists^\infty y R_i^{j,l}(d, y)$, uniformly in i, j, l .*

Proof. Using \mathbf{O}' , we proceed as follows: We first compute the number of e so that $R_i^{j,l}(a, e)$ for a generic a . Since the set of such e is \exists -definable in \mathcal{M}_k and finite, \mathbf{O}' can compute its size p , say. Then \mathbf{O}' computes the size of the set of d so that $\exists^{>p} y R_i^{j,l}(d, y)$. Again, this set is \exists -definable in \mathcal{M}_k and finite, so \mathbf{O}' can compute its size m , say. Finally, \mathbf{O}' can find all of the m elements $d \in M_k$ so that $\exists^{>p} y R_i^{j,l}(d, y)$. Among these, $\exists^\infty x R_i^{j,l}(z, x)$ holds if and only if at least two of $R_i^{j,l}(d, a)$, $R_i^{j,l}(d, b)$, and $R_i^{j,l}(d, c)$ hold for mutually generic elements a, b, c in \mathcal{M}_k . Since $R_i^{j,l}$ is \exists -definable, \mathbf{O}' can check whether these hold. \square

We can now prove some analogs of lemmas from the binary case.

Lemma 21. *Fix the model \mathcal{M}_k of T for $k \in [3, \omega]$ with a computable presentation in a relational language $\mathcal{L} = \{R_0, R_1, \dots\}$ such that the Morley rank of each relation is at most 1. Then for each generic a , $\text{Nbh}(a) = \text{iacl}(a)$ is a Σ_2^0 -subset of M_k ; in fact, \mathbf{O}' can compute the canonical index of the n -neighborhood of a , uniformly in any generic element a and $n \in \omega$. (In particular, the rank assumption can be made in the case of an at most ternary language by Lemma 18.)*

Proof. Each of the binary projections $R_i^{j,l}$ is \mathbf{O}' -computable. Andrews and Medvedev [AM14, Claim in the proof of Proposition 2.6] showed that if the language only consists of relations of rank at most 1, then $\text{iacl}(a)$ is exactly $\text{Nbh}(a)$.

Clearly, the second claim of our lemma implies the first, so fix n and a generic element a . The claim is trivial for $n = 0$, so assume we are given a canonical index for $\text{Nbh}_n(a)$. Fix any element $c \in \text{Nbh}_n(a)$. Since a and thus also c is generic, the second line of (*) cannot apply to c and any d . The first conjunct of the first line of (*) is Σ_1^0 and can apply to at most finitely many d , so \mathbf{O}' can effectively find all possible d . But then for each of these finitely many d , we can \mathbf{O}' -effectively check the second and third disjunct by Lemma 20. Thus \mathbf{O}' can compute a canonical index for $\text{Nbh}_{n+1}(a)$. \square

The counterpart of Lemma 21 for $\text{Nbh}(a)$ for $a \in \text{acl}(\emptyset)$ requires a more careful proof since now the second line of (*) can hold. However, we still have:

Lemma 22. *Fix the model \mathcal{M}_k of T with $k \in [3, \omega]$ with a computable presentation, say, in a relational language $\mathcal{L} = \{R_0, R_1, \dots\}$ such that the Morley rank of each relation is at most 1. Then $\text{acl}(\emptyset)$ is a Σ_2^0 -subset of \mathcal{M}_k . (In particular, the*

rank assumption can again be made in the case of an at most ternary language by Lemma 18.)

Proof. By Lemma 21, we can fix a generic element b and uniformly in \mathbf{O}' compute canonical indices of its n -neighborhoods. By Andrews and Medvedev [AM14, Proof of Proposition 2.6], $a \in \text{acl}(\emptyset)$ iff for some n , the n -neighborhoods of a and b are not isomorphic over $B_n := \{d \mid \exists i \leq n \exists^\infty y R_i^{j,l}(d, y)\}$. So we need to verify that \mathbf{O}' can enumerate all such a by enumerating enough of the neighborhood of a until we know that the n -neighborhoods of a and b are not isomorphic over B_n for some n . By Lemma 20, \mathbf{O}' can compute the set B_n . Clearly, the 0-neighborhoods of a and b are isomorphic, so fix n and assume we have $c \in \text{Nbh}_n(a)$. For any fixed R_i with $i \leq n+1$ of arity m , say, and any j, l with $1 \leq j < l \leq m$, first check, using Lemma 20, whether $\exists^\infty y R_i^{j,l}(c, y)$. If so, then $a \in \text{acl}(\emptyset)$. If not, then find the finite set of d so that $R_i^{j,l}(c, d)$ and $\neg \exists^\infty x R_i^{j,l}(x, d)$, or so that $\neg R_i^{j,l}(c, d)$ and $\exists^\infty x R_i^{j,l}(x, d)$ (again using Lemma 20). Doing this for all $c \in \text{Nbh}_n(a)$ and all i, j, l with $i \leq n+1$ and $1 \leq j < l \leq m$, we either find out that $a \in \text{acl}(\emptyset)$, or we compute a canonical index for $\text{Nbh}_{n+1}(a)$. \square

Corollary 23. *If T is a relational strongly minimal disintegrated theory such that each relation has rank at most 1, and if $k \in [3, \omega)$, then for each $a \in M_k$, $\text{iacl}(a)$ is Δ_2^0 . In particular, $\text{acl}(\emptyset)$ is a Δ_2^0 -set. (By Lemma 18, the rank assumption can always be made for ternary languages.)*

Proof. Let \mathcal{M}_k have basis B . Then M_k is the disjoint union $\text{acl}(\emptyset) \cup \bigcup_{b \in B} \text{iacl}(b)$. Each piece is Σ_2^0 by Lemmas 22 and 21; thus, each piece is Δ_2^0 . \square

We can now state and prove the main theorem of this section:

Theorem 24. *Let T be a strongly minimal disintegrated theory in a language \mathcal{L} consisting of relations $\{R_0, R_1, \dots\}$ of rank at most 1 (closed under permutations of variables) of bounded arity, and assume $k \in \text{SRM}(T)$ with $k \in [3, \omega)$. Then $[1, \omega] \subseteq \text{SRM}(T)$.*

Proof. We again proceed in two parts, the Up part and the Down part. For the entire proof of our theorem, we fix the set $B = \{d \mid \exists i, j, l \exists^\infty y R_i^{j,l}(d, y)\}$. We also fix three mutually generic elements a, b, c in \mathcal{M}_k .

Down: Fix $j \in [1, k)$.

Case 0: $\text{Nbh}(a)$ is finite. Then M_j is a cofinite, thus computable, subset of M_k .

Case 1: B is finite. In this case, $\text{Nbh}(g)$ is Σ_1^0 for any generic element g . Then M_j is a Δ_2^0 -subset of M_k by Corollary 23, and it contains the infinite Σ_1^0 -set $\text{Nbh}(a)$. Thus, by Lemma 10, \mathcal{M}_j has a computable copy.

Case 2: B is infinite. In this case, $\text{Nbh}(g)$ will generally only be Δ_2^0 for a generic element g by Corollary 23, but we can use the infinite Σ_1^0 -subset B of $\text{acl}(\emptyset)$. Again, M_j is a Δ_2^0 -subset of M_k containing an infinite Σ_1^0 -set B , so by Lemma 10, \mathcal{M}_j has a computable copy.

Note that in Case 2, in fact $[0, k] \subseteq \text{SRM}(T)$, as the argument works for $j = 0$ as well.

Up: We again work in two cases. In the first case, we will be able to find an infinite Σ_1^0 -set of disjoint tuples \bar{x} so that in each tuple, at least one element is generic. This will be useful in ensuring that what we build, in attempting to build a new generic neighborhood, is generic enough to handle further changes in our

approximations to $\text{Nbh}_n(g)$. In the second case, we will have that the neighborhood of a generic element is generated by a finite fragment of the language and thus is Σ_1^0 .

Case 1: For a generic element g , there are infinitely many disjoint \bar{x} so that

$$\exists i [R_i(g, \bar{x}) \wedge \neg \exists^\infty y R_i(y, \bar{x})].$$

For each tuple \bar{x} so that $R_i(g, \bar{x}) \wedge \neg \exists^\infty y R_i(y, \bar{x})$, at least one element in \bar{x} is in $\text{iacl}(g)$ since $g \in \text{acl}(\bar{x})$. Given that the three mutually generic elements a, b, c allow us to compute whether $\exists^\infty y R_i(y, \bar{x})$ holds for a tuple \bar{x} , this leads us to the following

Observation 25. *There is a Σ_1^0 -set C of tuples \bar{x} (of varying sizes) so that at least one of the elements in each tuple is a generic element and so that for each finite set X , there is a tuple $\bar{x} \in C$ disjoint from X . \square*

This observation allows us to conclude that if we see every $x \in \bar{x}$ realize some fixed \exists -formula, then we know that \exists -formula is generic. We want to “copy” $\text{iacl}(g)$ for some generic g in \mathcal{M}_k to build the model \mathcal{M}_{k+1} . The main difficulty is that $\text{iacl}(g)$ is a Δ_2^0 -subset of M_k . The source of movement out of $\text{iacl}(g)$ is observing an element being enumerated into B ; i.e., if the Δ_2^0 -approximation to the isomorphism type of $\text{Nbh}_n(g)$ changes, it is because some element of the current approximation to $\text{Nbh}_n(g)$ is enumerated into B .

At each stage, we will have copied a finite subset X of M_k and built an additional finite subset $\{d\} \cup Y$, where d is a new element intended to be generic over M_k , and where Y is, in the limit, a Π_1^0 -set, intended to be the interalgebraic closure of d . Elements in X come in three flavors:

- B_0 : These elements have already been enumerated into B .
- X_{ialg} : These elements currently appear to be in $\text{Nbh}_n(a) \cup \text{Nbh}_n(b) \cup \text{Nbh}_n(c)$ and we see some relations apparently witnessing this. Note that they may stop being in X_{ialg} if they appear currently in X_{ialg} due to some apparent connection to a , say, and some element on the path from a to the element moves into B , thus breaking the path.
- X_{free} : These are elements which currently appear to satisfy no relation with any elements of X_{ialg} , nor are they currently in B .

We let n be least so that the above partition exists. The only reason n might fail to give a partition as above at stage s is if there are elements $x, y \in X$ so that

$$x \in \text{Nbh}_n^s(a) \cup \text{Nbh}_n^s(b) \cup \text{Nbh}_n^s(c)$$

but

$$y \in (\text{Nbh}_{n+1}^s(a) \cup \text{Nbh}_{n+1}^s(b) \cup \text{Nbh}_{n+1}^s(c)) \setminus (\text{Nbh}_n^s(a) \cup \text{Nbh}_n^s(b) \cup \text{Nbh}_n^s(c)).$$

Thus n can be found and will be $\leq |X| + 1$. We also use a speed-up to ensure that at each stage s , the apparent neighborhoods $\text{Nbh}_n^s(a)$, $\text{Nbh}_n^s(b)$, and $\text{Nbh}_n^s(c)$ are isomorphic over B_0 .

We say that a quantifier-free \mathcal{L}_0 -diagram $\chi(X, dY)$ (for a finite fragment \mathcal{L}_0 of \mathcal{L}) is *allowed* at stage s if the following hold:

- There is an \mathcal{L}_0 -isomorphism over B_0 between $\{d\} \cup Y$ and a subset A of $\text{Nbh}_n^s(a)$.
- For each tuple $\bar{x} \subseteq X$ intersecting $X_{\text{free}} \cup X_{\text{ialg}}$ nontrivially, each tuple \bar{y} in $Y \cup \{d\}$, and each $R \in \mathcal{L}_0$, $\neg R(\bar{x}, \bar{y})$ holds.

- There is a tuple $\bar{x} \in C$ (from Observation 25) such that for each $x \in \bar{x} \setminus B^s$, $\text{Nbh}_n^s(x)$ is disjoint from X , no elements in $\text{Nbh}_n^s(x)$ realize any relations with $X \setminus B_0$, and some $A' \subseteq \text{Nbh}_n^s(x)$ satisfies $\chi(X, -)$. (Note, in particular, that there is an \mathcal{L}_0 -isomorphism h_x of $\{d\} \cup Y$ with A' over all of X .)

In building the new generic element d and its interalgebraic closure, we will ensure that at all stages, we have committed to an allowed diagram. The idea is that if our approximations to $\text{Nbh}_n^s(a)$, $\text{Nbh}_n^s(b)$ and $\text{Nbh}_n^s(c)$ change by elements entering B , then there are elements in $\text{Nbh}_n^s(x)$ for a generic element $x \in X$ which can be used as images of the corresponding elements in $\{d\} \cup Y$.

We need to verify the following claims:

Claim 26. *Suppose χ is an allowed \mathcal{L}_0 -diagram and $m \leq n$, and fix $x \in \bar{x} \setminus B^s$. If $\text{Nbh}_m^s(a)$, $\text{Nbh}_m^s(b)$, and $\text{Nbh}_m^s(c)$ are correct, then the elements of $\{d\} \cup Y$ in $h_x^{-1}(\text{Nbh}_m^s(x))$ are \mathcal{L}_0 -isomorphic to a subset of $\text{Nbh}_m(g)$ for an element g generic over M_k .*

Proof. Fix an element $x \in \bar{x}$. Since $\{d\} \cup Y$ is \mathcal{L}_0 -isomorphic to A' over X , it suffices to see that $A' \cap \text{Nbh}_m^s(x)$ is \mathcal{L}_0 -isomorphic over X to a subset of $\text{Nbh}_m(g)$. There is a natural \mathcal{L}_0 -isomorphism f from $\text{Nbh}_m^s(x)$ to $\text{Nbh}_m(g)$ over B_0 . It remains to see that this is an \mathcal{L}_0 -isomorphism over X . Given a tuple $\bar{c}\bar{d}$, where $\bar{c} \in X$, $\bar{d} \in A' \cap \text{Nbh}_m^s(x)$, and $R \in \mathcal{L}_0$, it is assumed that $\neg R(\bar{c}, \bar{d})$ if $\bar{c} \not\subseteq B_0$. In that case, $\neg R(\bar{c}, f(\bar{d}))$ also holds, as otherwise $\bar{c} \subseteq B_0$. Note that no element of X_{free} can move into B_0 witnessed by a relation in \mathcal{L}_0 , as that would witness a change in $\text{Nbh}_n(a)$.

If $\bar{c} \subseteq B_0$, then since f is a \mathcal{L}_0 -isomorphism over B_0 , we have $R(\bar{c}, \bar{d})$ if and only if $R(\bar{c}, f(\bar{d}))$. \square

Claim 27. *If χ_0 is an allowed \mathcal{L}_0 -diagram which gives the \mathcal{L}_0 -diagram of a subset of $\text{Nbh}(g)$ over X , then for the true quantifier-free \mathcal{L}_0 -type χ of $\text{Nbh}_n(g)$ over X (which extends χ_0), there will be a stage at which χ is allowed, where $h_x^{-1}(\text{Nbh}_n^s(x))$ is isomorphic to $\text{Nbh}_n(g)$ over $X_{\text{free}} \cup B_0$ for an element g generic over M_k .*

Proof. Let χ be the true quantifier-free \mathcal{L}_0 -type of $\text{Nbh}_n(g)$ over X . Then this is consistent with χ_0 by assumption. Since it is generic, any sufficiently generic element will satisfy the existence of such an n -neighborhood satisfying χ . Using Observation 25, we find a tuple \bar{x} where each $x \in \bar{x}$ has the needed neighborhood. This exists as, after some finite search, we find tuples \bar{x} where each element is generic enough. Then, after some finite search, we see the necessary neighborhoods of these elements. \square

Claim 28. *If χ is an allowed diagram and some elements are removed from one of $\text{Nbh}_n^s(a)$, $\text{Nbh}_n^s(b)$, or $\text{Nbh}_n^s(c)$ (due to enumeration into B) and move into B_0 or into X_{free} , then there is an effective way to identify the corresponding elements in Y with elements in $\text{Nbh}_n^s(x)$ for some $x \in \bar{x}$ so that after this identification, there is a larger X (i.e., X has grown to include these old elements of Y) such that we once again have an allowed configuration.*

Proof. Let t be a stage so large that $\text{Nbh}_n^t(a)$, $\text{Nbh}_n^t(b)$, $\text{Nbh}_n^t(c)$ are all correct, as is $\text{Nbh}_n^t(x)$ for a generic $x \in \bar{x}$. Let Z be the set of elements removed from $\text{Nbh}_n^s(x)$ by stage t , and let Z' be the set of corresponding elements from $\{d\} \cup Y$. We need to verify only that then χ_0 is a fragment of the true type of $\text{Nbh}_n(g)$ over $X \cup Z$.

Once we have this, the fact that we find an allowed configuration extending it is proved above. Note that after identifying Z with Z' , we have that Y is isomorphic to a subset of $\text{Nbh}_n^t(x)$ over $X \cup Z$. In particular, the only relations holding between $\text{Nbh}_n^t(x)$ and $X \cup Z$ are between $\text{Nbh}_n^t(x)$ and B_0^t . Thus the quantifier-free type of $dY \setminus Z'$ over $X \cup Z$ is a fragment of the type of the neighborhood of a generic element, and thus we will eventually see an allowed configuration extending it by Claim 27, where $\chi_0 = \emptyset$. \square

Using the claims above, we give the construction of \mathcal{M}_{k+1} . We copy M_k , one element and relation at a time. At each stage, we search to find an allowed diagram χ , and we build elements $\{d\} \cup Y$ in addition to $X \subset M_k$. At each stage $s \geq m$, we have an approximation $\text{Nbh}_m^s(g)$ for the m -neighborhood of a generic element g . We use Claim 27 to know that we can seek an allowed diagram which contains a copy of this entire $\text{Nbh}_m^s(g)$ (that is, unless our approximation changes, which is also considered a success). Whenever our approximations to $\text{Nbh}_m^s(g)$ changes (thus $\text{Nbh}_m^s(a)$, $\text{Nbh}_m^s(b)$, and $\text{Nbh}_m^s(c)$ change), we use Claim 28 to return to an allowed configuration. Now we need to see that in the limit, we build $M_k \cup \text{Nbh}(g)$. Consider a stage t late enough that $\text{Nbh}_m^t(a)$, $\text{Nbh}_m^t(b)$, and $\text{Nbh}_m^t(c)$ have all settled down. By Claim 27, we succeed at stage t to extend to an allowed configuration where the full m -neighborhood of g is represented. As $\text{Nbh}_m^t(a)$, $\text{Nbh}_m^t(b)$, and $\text{Nbh}_m^t(c)$ do not change, there is no more injury, and by Claim 26 we commit to the correct $\text{Nbh}_m(g)$. Thus, for each m , from some stage onward, we have copied the correct $\text{Nbh}_m(g)$. Thus we build a copy of \mathcal{M}_{k+1} .

This suffices to show in Case 1 that $[1, \omega] \subseteq \text{SRM}(T)$. To see that $\omega \in \text{SRM}(T)$, we run the previous construction, building more and more generic neighborhoods. For a configuration to be allowed when building l many new neighborhoods, we need l many different tuples $\bar{x} \in C$ as protection. Everything else remains the same.

Case 2: For a generic element g , there are only finitely many disjoint tuples \bar{x} so that

$$\exists i [R_i(g, \bar{x}) \wedge \neg \exists^\infty y R_i(y, \bar{x})].$$

Fix h_1, \dots, h_n so that $\forall \bar{x} (R(g, \bar{x}) \rightarrow \bar{x} \cap \{h_1, \dots, h_n\} \neq \emptyset)$. Fix s so that for each h_l ($l \leq n$), there is an $i \leq s$ so $R_i^{12}(g, h_l)$. Let N_i for $i \leq s$ be the number of tuples \bar{y} so that $R_i(g, \bar{y})$. Finally, we consider the new language

$$\mathcal{L}' := \{R_i^{12} \mid i \leq s\} \cup \{\exists y (R_i^{12}(g, y) \wedge R_n(g, y, \bar{x})) \mid i \leq s \text{ and } n \in \omega\}.$$

Now, note that using the (finitely many, non-uniformly fixed) numbers N_i , the relations in \mathcal{L}' are uniformly computable on \mathcal{M} . Furthermore, the neighborhood of g in \mathcal{M} is the same in the two languages \mathcal{L} and \mathcal{L}' . The only difference is that now \mathcal{L}' is a relational language of strictly smaller arity than \mathcal{L} .

We again consider whether Case 1 holds for \mathcal{L}' . If so, the proof proceeds as above; in particular, Observation 25 holds, and the proof in Case 1 (using the language \mathcal{L} ; we use \mathcal{L}' only to make Observation 25 true) works as before. If not, this reduction, after finitely many steps (since our language has bounded arity), gets us down to the case where \mathcal{L}' will be a finite binary relational language.

In that case, for each generic element g in \mathcal{M}_k , $\text{Nbh}_n(g)$ is a uniformly computable subset of M_k . Thus $\text{iacl}(g)$ is a Σ_1^0 -subset of M_k . We build a copy of \mathcal{M}_{k+1} (in the original language \mathcal{L}) by copying M_k along with A , a copy of $\text{iacl}(g)$ for a generic element g . We need to determine, for each tuple $\bar{c} \in M_k \cup A$ and each i ,

whether $R_i(\bar{c})$ holds. If \bar{c} is contained in M_k or in A , then we copy the corresponding fact. Otherwise, write $\bar{c} = \bar{c}_0\bar{c}_1$ where $\bar{c}_0 \in M_k$ and $\bar{c}_1 \in A$. Let \bar{g}_1 be the elements in $\text{acl}(g)$ which we are copying to make \bar{c}_1 , and let g' be an element generic over g . We search for an n so that $\bar{g}_1 \in \text{Nbh}_n(g)$. Then we say $R_i(\bar{c})$ holds if and only if $\mathcal{M}_k \models R_i(\bar{c}_0, \bar{g}_1)$ and $\mathcal{M}_k \models R_i(\bar{c}_0, \bar{g}'_1)$ for some $\bar{g}'_1 \subseteq \text{Nbh}_n(g')$. Note that in this case \bar{c}_0 must be in $\text{acl}(\emptyset)$ since R has rank ≤ 1 . So $R(\bar{c}_0, \bar{g}_1)$ implies $R(\bar{c}_0, \bar{c}_1)$ in \mathcal{M}_{k+1} . If no tuple $\bar{g}'_1 \in \text{Nbh}_n(g')$ satisfies $R_i(\bar{c}_0, \bar{g}'_1)$, then \bar{c}_0 is not in $\text{acl}(\emptyset)$ (as there is an isomorphism of \mathcal{M} fixing $\text{acl}(\emptyset)$ mapping g to g'), and thus $R(\bar{c}_0, \bar{c}_1)$ cannot hold in \mathcal{M}_{k+1} , as this would make the rank of R be ≥ 2 . Thus we have built a copy of \mathcal{M}_{k+1} .

We finally note that in Case 2, we obtained \mathcal{M}_{k+1} from \mathcal{M}_k uniformly with M_k as a computable subset of M_{k+1} , and so we can also build a computable copy of \mathcal{M}_ω . This concludes the proof of Theorem 24. \square

Corollary 29. *For any strongly minimal disintegrated T in an at most ternary relational language, if $\text{SRM}(T) \cap [3, \omega] \neq \emptyset$, then $[1, \omega] \subseteq \text{SRM}(T)$.*

Proof. Fix a computable copy \mathcal{M} of the model \mathcal{M}_l of T for some $l \in [3, \omega]$. By Lemma 18, we have a computable presentation of \mathcal{M}' , the structure \mathcal{M} seen as an \mathcal{L}' -structure, where \mathcal{L}' is given by Lemma 18. For each $k \in [1, \omega]$, Theorem 24 gives a computable copy \mathcal{N}' of \mathcal{M}_k in the language \mathcal{L}' . Since each relation symbol in \mathcal{L} is uniformly a Boolean combination of relation symbols in \mathcal{L}' , we obtain a computable copy \mathcal{N} of \mathcal{M}_k . \square

This concludes the proof of Theorem 14. \square

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