

Comparing classes of finite sums

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Abstract

The notion of *Turing computable embedding* [4] provides an effective way to compare classes of countable structures, reducing the classification problem for one class to that for the other. Most of the known results on non-existence of Turing computable embeddings reflect differences in the complexity of the sentences needed to distinguish among non-isomorphic members of the two classes. Here we give some examples of further distinctions that we can make using Turing computable embeddings. The classes that we consider consist of sum structures. We consider cardinal sums of n structures, in which the components are named by predicates, and sums given by an equivalence relation, where the components are not named. We also consider direct sums of certain groups. The results are based on model-theoretic considerations related to Morley degree. The proofs of non-embeddability involve index set calculations.

1 Introduction

We consider classes K consisting of structures all having the same computable language, and all with universe a subset of ω . Our classes are closed under isomorphism. We say that a class is *nice* if it is axiomatized by a computable infinitary sentence. If we are interested just in computable structures, then we may accept as nice a class for which the set of computable indices is hyperarithmetical—this is equivalent to saying that there is a computable infinitary sentence whose computable models are just the computable members of our class. Here is the main definition from [4].

Definition 1. *A Turing computable embedding of K into K' is a Turing operator $\Phi = \varphi_e$ such that the following hold:*

1. *For each $\mathcal{A} \in K$, there is some $\mathcal{B} \in K'$ such that $\varphi_e^{D(\mathcal{A})} = \chi_{D(\mathcal{B})}$ —we write $\Phi(\mathcal{A})$ for \mathcal{B} .*

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2. For $\mathcal{A}, \mathcal{A}' \in K$, $\mathcal{A} \cong \mathcal{A}'$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{A}')$.

We write $K \leq_{tc} K'$ if there is a Turing computable embedding of K into K' . The relation \leq_{tc} is a pre-ordering on classes of structures. The notion of Borel embedding, introduced in [8], gives a pre-ordering \leq_B , on classes of structures with universe ω . Using \leq_{tc} , we can make some distinctions that are not made using \leq_B . Under \leq_B , all classes with \aleph_0 isomorphism types are equivalent. Under \leq_{tc} , we have

$$NF <_{tc} V$$

where NF is the class of number fields, and V is the class of non-trivial \mathbb{Q} -vector spaces. The class V is tc -equivalent to the class of free groups. The class LO lies on top, along with the class UG of undirected graphs. The class ApG of Abelian p -groups does not lie on top [7].

For \leq_{tc} , many of the known results have been discovered by considering the kinds of sentences that distinguish among non-isomorphic members of the classes and applying the following result from [11].

Theorem 1 (Pullback Theorem). *Suppose $K \leq_{tc} K'$ via Φ . Then for any computable infinitary sentence φ in the language of K' , we can find a sentence φ^* in the language of K such that for $\mathcal{A} \in K$, $\mathcal{A} \models \varphi^*$ iff $\Phi(\mathcal{A}) \models \varphi$. Moreover, if φ is computable Σ_α , where α is a computable ordinal ≥ 1 , then φ^* is also computable Σ_α .*

As an example of the use of the Pullback Theorem, we show that $V \not\leq_{tc} NF$ (see [12]).

Sample proof. Suppose $V \leq_{tc} NF$ via Φ , expecting a contradiction. Let $\mathcal{A}, \mathcal{A}'$ be non-isomorphic \mathbb{Q} -vector spaces. Then $\Phi(\mathcal{A}), \Phi(\mathcal{A}')$ are non-isomorphic number fields. The number fields must differ on some existential sentence φ . The Pullback Theorem gives a computable Σ_1 sentence φ^* on which $\mathcal{A}, \mathcal{A}'$ should differ. However, all elements of V satisfy the same computable Σ_1 sentences [11]. \square

in [7], the Pullback Theorem is used for a general result implying that $UG \not\leq_{tc} ApG$. There are non-isomorphic undirected graphs G_1, G_2 such that $\omega_1^{G_i} = \omega_1^{G_j}$ and G_1, G_2 satisfy the same computable infinitary sentences, but non-isomorphic Abelian p -groups that compute no non-computable ordinals must differ on some computable infinitary sentence.

In the present paper, we make some further distinctions, based on model theoretic differences related to Morley degree, not complexity of sentences. We consider classes of sum structures with a fixed number of summands.

Definition 2. *Let K be a class of structures, closed under isomorphism.*

1. K^n is the class of cardinal sums of n elements of K , where the components of the cardinal sum are named by disjoint unary predicates,

2. K^{n*} is the class of structures consisting of an equivalence relation with n equivalence classes, with an element of K on each equivalence class.

We will show that if V is the class of non-trivial \mathbb{Q} -vector spaces, then $V^n <_{tc} V^{n+1}$. We will also show that $V^n \equiv_{tc} V^{n*}$. It follows that $V^{n*} <_{tc} V^{(n+1)*}$. Similarly, if S is the class of sets, with no relations, we show that $S^n <_{tc} S^{n+1}$ and $S^n \equiv_{tc} S^{n*}$, so $S^{n*} <_{tc} S^{(n+1)*}$. We consider also certain classes of orderings and trees. Finally, we consider direct sums of certain groups. Let P^n be the class of direct sums of n Abelian p -groups, of the form Z_{p^n} or Z_{p^∞} . We show that $P^n <_{tc} P^{n+1}$ by showing that $P^n \equiv_{tc} S^n$. The proofs of non-embeddability use index set calculations.

1.1 Index sets

We identify a structure \mathcal{A} with its atomic diagram. Thus, \mathcal{A} is *computable* if there is some e such that $\varphi_e = \chi_{D(\mathcal{A})}$.

Definition 3.

1. For a structure \mathcal{A} , the index set $I(\mathcal{A})$ is the set of indices for computable copies of \mathcal{A} .
2. For a class K , $I(K)$ is the set of indices for computable members of K .

Definition 4 (Nice class). *Suppose K is a class of structures closed under isomorphism. We say that K is nice if it is axiomatized by a computable infinitary sentence.*

Note that if K is a nice class, then $I(K)$ is hyperarithmetical.

Definition 5 (Γ -hard, Γ -complete). *Let Γ be a complexity class (such as Π_3^0 or Σ_1^1), and let $A \subseteq \omega$.*

1. A is Γ -hard if for every $S \in \Gamma$, $S \leq_m A$,
2. A is m -complete Γ if $A \in \Gamma$, and A is Γ -hard.

The following index set calculations are given in [3].

Example 1. *Recall that V is the class of non-trivial \mathbb{Q} -vector spaces. Let $\mathcal{A} \in V$.*

1. If \mathcal{A} has infinite dimension, then $I(\mathcal{A})$ is m -complete Π_3^0 .
2. If \mathcal{A} has dimension 3, then $I(\mathcal{A})$ is m -complete d - Σ_2^0 ; i.e., it is a difference of Σ_2^0 sets.

Sometimes, we want to describe a specific structure \mathcal{A} so as to differentiate it from other members of a class K . The description of K may be complicated, more complicated than the description of \mathcal{A} within K . The following definitions are from [2].

Definition 6 (Γ within, Γ -hard within, Γ -complete within). *Let $A, B \subseteq \omega$, where $A \subseteq B$.*

1. *A is Γ within B if there is some $C \in \Gamma$ such that $A = C \cap B$.*
2. *A is Γ -hard within B if for each $S \in \Gamma$, there a computable function f such that for all n , $f(n) \in B$, and $f(n) \in A$ iff $n \in S$.*
3. *A is m -complete Γ within B if it is Γ within B , and Γ -hard within B .*

2 Cardinal sums

In this section, we consider cardinal sums of sets, cardinal sums of vector spaces, and cardinal sums of structures from some classes that are not nice.

2.1 Sums of sets

Let S be the class of structures that are just sets. Then S^n is the class of cardinal sums of n elements of S .

Proposition 2. $S^n <_{tc} S^{n+1}$

Proof. There is an obvious embedding Φ witnessing that $S^n \leq_{tc} S^{n+1}$. For each input structure \mathcal{A} , $\Phi(\mathcal{A}) = \mathcal{B}$, where the first n components of \mathcal{B} have the same size as the corresponding components of \mathcal{A} , and the last component of \mathcal{B} has a fixed size, say it is infinite. We must show that $S^{n+1} \not\leq_{tc} S^n$. It is enough to prove the following.

Lemma 3. *If $S^s \leq_{tc} S^t$, via Φ , then Φ takes a structure with r infinite sets to one with at least r infinite sets.*

Proof. We proceed by induction. For $r = 1$, let u be a computable member of S^s with just one infinite set, and let v be a computable member of S^t with no infinite sets. Suppose $\Phi(u) = v$, expecting a contradiction. From Φ , we derive a partial computable function $f : I(S^s) \rightarrow I(S^t)$ such that f takes $I(u)$ to $I(v)$.

Claim 1: $I(v)$ is d -c.e. within $I(S^t)$. Since v is a finite structure, we have a finitary d -c.e. description. If m is the size of v , we say that there are m elements located in the appropriate sets, and there do not exist $m + 1$ elements.

Claim 2: $I(u)$ is m -complete Π_2^0 within $I(S^s)$. We have a computable Π_2 description giving the sizes of all s sets, including the infinite one. Therefore, $I(u)$ is Π_2^0 . To show hardness, let S be a Π_2^0 set. We have a uniformly computable sequence $(\mathcal{C}_k)_{k \in \omega}$ of structures in S^s such that

- if $k \in S$, then $\mathcal{C}_k \cong u$, and
- if $k \notin S$, then the components of u match those of \mathcal{C}_k except that the infinite set in u corresponds to a finite set in \mathcal{C}_k .

For $e \in I(S^s)$, $f(e)$ is always in $I(S^t)$, and $e \in I(u)$ iff $f(e) \in I(v)$. Therefore, $I(u)$ is d -c.e. within $I(S^s)$, a contradiction. We have proved the lemma for the case where $r = 1$.

Assuming that the lemma holds for r , we prove it for $r + 1$. Let u be a computable member of S^s with $r + 1$ infinite sets, and let v be a computable member of S^t with just r infinite sets. Suppose $\Phi(u) = v$, expecting a contradiction. Let L^s consist of the elements of S^s with at least r infinite sets, and let L^t consist of the elements of S^t with at least r infinite sets. From Φ , we derive a partial computable function $f : I(S^s) \rightarrow I(S^t)$ such that f takes $I(u)$ to $I(v)$ and takes $I(L^s)$ to $I(L^t)$.

Claim 3: $I(v)$ is d -c.e. within $I(L^t)$. We can describe v within L^t by saying how many elements there are in the $t - r$ sets that are finite.

Claim 4: $I(u)$ is m -complete Π_2^0 within $I(L^t)$. We can describe u by giving the sizes of all sets. Therefore, $I(u)$ is Π_2^0 within $I(S^s)$ and also within $I(L^s)$. To show that $I(u)$ is m -complete Π_2^0 within $I(L^s)$, let R be a Π_2^0 set. We have a uniformly computable sequence $(\mathcal{C}_k)_{k \in \omega}$ of elements of L^s such that each component of \mathcal{C}_k has the same size as the corresponding component of u except that the component of \mathcal{C}_k corresponding to the last infinite component of u is infinite if $k \in R$ and finite otherwise.

For $e \in I(L^s)$, $f(s)$ is always in $I(L^t)$, and $e \in I(u)$ iff $f(e) \in I(v)$. Therefore, $I(u)$ is d -c.e. within $I(L^s)$. This contradicts Claim 4 above. \square

\square

2.2 Sums of vector spaces

Recall that V is the class of non-trivial \mathbb{Q} -vector spaces. Then V^n is the class of cardinal sums of n vector spaces.

Proposition 4. $V^n <_{tc} V^{n+1}$.

Proof. There is an obvious embedding—we make the first n components of the output match those of the input, and for the last component of the output, we fix the dimension, say it is infinite. To prove that $V^{n+1} \not\leq_{tc} V^n$, it is enough to show the following.

Lemma 5. *If Φ is an embedding of V^s into V^t , then Φ takes a tuple with r components of infinite dimension to one with at least r components of infinite dimension.*

Proof. We proceed by induction on r . For $r = 1$, let u be a computable member of V^s with one component of infinite dimension, and let v be a computable member of V^t with no components of infinite dimension. Suppose $\Phi(u) = v$, expecting a contradiction. From Φ , we get a partial computable function f from $I(V^s)$ to $I(V^t)$ taking $I(u)$ to $I(v)$.

Claim 1: $I(v)$ is $d\text{-}\Sigma_2^0$ within $I(V^t)$. We have a computable $d\text{-}\Sigma_2$ description of v , saying for each component that the dimension is at least some k , and not at least $k + 1$.

Claim 2: $I(u)$ is m -complete Π_3^0 within $I(V^s)$. We have a computable Π_3^0 Scott sentence for u , giving the dimensions of all components. Therefore, $I(u)$ is Π_3^0 . To show hardness, let S be a Π_3^0 set. We have a uniformly computable sequence \mathcal{C}_n of structures in V^s such that if $n \in S$, then $\mathcal{C}_n \cong u$, and if $n \notin S$, then for any component of u of finite dimension, the corresponding component of \mathcal{C}_n has the same dimension, and for the component of u of infinite dimension, the corresponding component of \mathcal{C}_n has finite dimension.

For $e \in I(V^s)$, $f(e)$ is always in $I(V^t)$, and $e \in I(u)$ iff $f(e) \in I(v)$. Therefore, $I(u)$ is $d\text{-}\Sigma_2^0$ within $I(V^s)$. This contradicts Claim 2. Therefore, the lemma holds for $r = 1$.

Assuming that the lemma holds for r , we prove it for $r + 1$. Let u be a computable member of V^s with $r + 1$ components of infinite dimension, and let v be a computable member of V^t with just r components of infinite dimension. Suppose $\Phi(u) = v$, expecting a contradiction. Let L^s consist of the elements of V^s with at least r components of infinite dimension, and let L^t consist of the elements of V^t with at least r components of infinite dimension. From Φ we derive a partial computable function f that maps $I(V^s)$ to $I(V^t)$, taking $I(u)$ to $I(v)$, and taking $I(L^s)$ to $I(L^t)$.

Claim 3: $I(v)$ is $d\text{-}\Sigma_2^0$ within $I(L^t)$. We can describe v within L^t by saying, for each of the $t - r$ finite dimensional components, that the dimension is at least some k and not at least $k + 1$.

Claim 4: $I(u)$ is m -complete Π_3^0 within $I(L^t)$. We have a computable Π_3 Scott sentence for u , giving the dimensions of all components. Therefore, $I(u)$ is Π_3^0 . We show that $I(u)$ is m -complete Π_3^0 within L^s . Let S be a Π_3^0 set., We have a uniformly computable sequence $(\mathcal{C}_k)_{k \in \omega}$ of elements of L^s such that each component of \mathcal{C}_k has the same dimension as the corresponding component of u , except possibly the component corresponding to the last in u of infinite dimension. This has infinite dimension if $k \in S$ and finite dimension otherwise.

For $e \in I(L^s)$, $f(s)$ is always in $I(L^t)$, and $e \in I(u)$ iff $f(e) \in I(v)$. Therefore, $I(u)$ is $d\text{-}\Sigma_2^0$ within $I(L^s)$, contradicting Claim 4 above. \square

\square

2.3 Sums of ordinals, rank-saturated trees, etc.

Let K be a class of structures such that among the computable members, there is just one (up to isomorphism) for which the index set is not hyperarithmetical. Here are some examples:

Example 2. Let K be the class of orderings that are either well orderings or have the form $\alpha(1 + \eta)$, for an admissible ordinal α .

The next example is a special class of trees. For a node σ in a tree $T \subseteq \omega^{<\omega}$, we write $tr(\sigma)$ for the *tree rank*, or *foundation rank* of the node σ . Recall that $tr(\sigma) = 0$ if σ has no successors, for $\alpha > 0$, $tr(\sigma) = \alpha$ if all successors of σ have ordinal ranks, and α is the least ordinal greater than all of these, and $tr(\sigma) = \infty$ if σ does not have ordinal rank.

Example 3. Let K be the class of trees $T \subseteq \omega^{<\omega}$ such that, for the least α that is the ordinal of an admissible set containing a copy of T , for all $\sigma \in T$, either

1. $tr(\sigma)$ is an ordinal less than α and for all $\beta < \alpha$, σ has infinitely many successors σ' with $tr(\sigma') = \beta$, or
2. $tr(\sigma) = \infty$ and for all $\beta < \alpha$, σ has infinitely many successors σ' with $tr(\sigma') = \beta$, plus infinitely many successors σ' with $tr(\sigma') = \infty$

The two classes above are not nice. We do not know that there is a nice class in which just one computable member (up to isomorphism) has non-hyperarithmetical index set. Assuming that Vaught's Conjecture fails, we get a relativized version of such a class. Becker [1] has shown that if there is a counterexample to Vaught's Conjecture, then there is an $L_{\omega_1\omega}$ sentence φ such that for a cone of sets X , φ has, up to isomorphism, a unique X -computable model for which the set of X -computable indices is not X -hyperarithmetical; i.e., it is not in the least admissible set containing X .

Proposition 6. Suppose K is a class with (up to isomorphism) just one computable member \mathcal{A} for which the index set is non-hyperarithmetical. Then $K^n <_{tc} K^{n+1}$.

Proof. The proof follows the same outline as the earlier results. There is an obvious embedding witnessing that $K^n \leq_{tc} K^{n+1}$. We copy the first n components of the input structure, and we fix the last component, giving it type \mathcal{A} . To show that $K^{n+1} \not\leq_{tc} K^n$, it is enough to prove the following.

Lemma 7. If $K^s \leq_{tc} K^t$ via Φ , then Φ maps computable structures with r components of type \mathcal{A} to structures with at least r components of type \mathcal{A} .

Proof. We proceed by induction on r . For $r = 1$, let u be a computable member of K^s with one entry of type \mathcal{A} , and let v be a computable member of K^t with no entries of type \mathcal{A} . We have the following.

Claim 1: $I(v)$ is hyperarithmetical. For each $\mathcal{B} \in K$ such that $\mathcal{B} \cong \mathcal{A}$, $I(\mathcal{B})$ is hyperarithmetical. By a result in [9], there is a computable infinitary sentence φ whose computable models are just those isomorphic to \mathcal{A} . Using such sentences, we obtain a computable infinitary sentence whose computable models are just the copies of v . If this sentence is computable Σ_α , then $I(v)$ is Σ_α^0 .

Claim 2: We have a Turing operator Ψ taking K to K^s such that in the output structure, a copy of the input structure fills the place of \mathcal{A} in u , and the other components of the output are as for u . From Ψ , we get a partial computable function g taking $I(K)$ to $I(K^s)$ such that $e \in I(\mathcal{A})$ iff $g(e) \in I(u)$. Then $I(u)$ is not hyperarithmetical within $I(K)^s$.

From Φ , we get a partial computable function mapping $I(K^s)$ to $I(K^t)$ such that $e \in I(u)$ iff $f(e) \in I(v)$. Therefore, $I(u)$ is hyperarithmetical within $I(K^s)$. This contradicts Claim 2. Therefore, the lemma holds for $r = 1$.

Assuming that the lemma holds for r , we prove it for $r + 1$. Let u be a computable member of K^s with $r + 1$ components of isomorphism type \mathcal{A} , and let v be a computable member of K^t with just r components of type \mathcal{A} . Suppose $\Phi(u) = v$, expecting a contradiction. Let L^s consist of the elements of K^s with at least r components of type \mathcal{A} , and let L^t consist of the elements of K^t with at least r components of type \mathcal{A} . From Φ we get a partial computable function f from $I(K^s)$ to $I(K^t)$, taking $I(u)$ to $I(v)$ and taking $I(L^s)$ to $I(L^t)$.

Claim 3: $I(v)$ is hyperarithmetical within $I(L^t)$. We describe v within L^t by describing $t - r$ components not of type \mathcal{A} .

Claim 4: $I(u)$ is not hyperarithmetical within $I(L^s)$. Let Ψ be a Turing operator taking K to K^s such that in the output structure, a copy of the input structure fills the last place filled by \mathcal{A} in u , and the other components of the output are the same as for u . This gives a partial computable function g taking $I(K)$ to $I(L^s)$ such that $e \in I(\mathcal{A})$ iff $g(e) \in I(u)$. Then $I(u)$ is not hyperarithmetical within $I(L^s)$.

From Φ , we get a partial computable function f taking $I(K^s)$ to $I(K^t)$, such that f takes $I(L^s)$ to $I(L^t)$, and $e \in I(u)$ iff $f(e) \in I(v)$. Therefore, $I(u)$ is hyperarithmetical within $I(L^s)$, contradicting Claim 4 above. \square

\square

2.4 \leq_{FF} -reducibility

In the results so far, the proofs of non-existence of Turing computable embeddings involved computable structures and index sets. For a class K , let $E(K)$ be the set of pairs $(a, b) \in \omega^2$ such that either a and b are indices for computable members of K that are isomorphic, or else a and b are both not in $I(K)$. If K is a nice class, then $E(K)$ is a Σ_1^1 equivalence relation on ω . Friedman and Fokina [5] defined a variant of Turing computable embedding for comparing Σ_1^1 equivalence relations on ω .

Definition 7. Let E, E' be equivalence relations on ω . We write $E \leq_{FF} E'$ if there is a computable function $f : \omega \rightarrow \omega$ such that mEn iff $f(m)E'f(n)$. If $E = E(K)$ and $E' = E(K')$, where K and K' are nice classes of structures, then we write $K \leq_{FF} K'$.

In [6], it is shown that isomorphism on computable trees lies on top under \leq_{FF} . The same is true for isomorphism of computable torsion-free Abelian groups (not known to be on top under \leq_{tc}) and even Abelian p -groups (known not to be on top under \leq_{tc}).

We may define a variant of \leq_{FF} , on equivalence relations that are not defined on all of ω . In the cases that interest us, the equivalence relation E is Σ_1^1 within $S \times S$, where S is the field of E . Suppose E, E' are equivalence relations, defined on sets $S, S' \subseteq \omega$. We write $E \leq_{FF} E'$ if there is a partial computable function f , such that

1. f maps S into S' (the domain of f may be larger than S), and
2. for $m, n \in S$, mEn iff $f(m)E'f(n)$.

We note the following simple fact.

Remark. If K, K' are classes of structures (closed under isomorphism, but not necessarily nice), and $K \leq_{tc} K'$, then $E(K) \leq_{FF} E(K')$.

Above, we gave two examples of classes K such that among the computable members, there is (up to isomorphism) just one with non-hyperarithmetical index set. The computable members of the two classes are as follows:

1. the computable linear orderings that have type α for some computable ordinal α , or have Harrison type $\omega_1^{CK}(1 + \eta)$,
2. the computable rank-saturated trees—if $tr(\sigma) = \infty$, then there are infinitely many unranked successors, and infinitely many of rank β for each computable ordinal β .

Our proofs yield the following.

Theorem 8. *For any of the following classes K , we have $E(K^n) <_{FF} E(K^{n+1})$:*

1. S —the class of sets
2. V —the class of \mathbb{Q} -vector spaces
3. computable well orderings, plus orderings of Harrison type,
4. computable rank-saturated trees.

3 Sums over equivalence classes

Recall that for a class K , K^{n*} consists of structures with an equivalence relation that partitions the universe into n equivalence classes, with a structure from K on each equivalence class. Recall that S is the class of sets, and V is the class of non-trivial \mathbb{Q} -vector spaces.

Proposition 9.

1. $S^{n*} <_{tc} S^{(n+1)*}$
2. $V^{n*} <_{tc} V^{(n+1)*}$

We know that $S^n <_{tc} S^{n+1}$ and $V^n <_{tc} V^{n+1}$. Therefore, to prove Proposition 9, it is enough to prove that $S^n \equiv_{tc} S^{n*}$ and $V^n \equiv_{tc} V^{n*}$.

Lemma 10. $S^n \equiv_{tc} S^{n*}$

Proof. We first show that $S^n \leq_{tc} S^{n*}$. Given an input structure $\mathcal{A} \in S^n$, we produce an output structure $\mathcal{B} \in S^{n*}$ that codes which components of \mathcal{A} have which size. When we are building \mathcal{B} , there is an ordering on the classes, corresponding to that in \mathcal{A} . If at stage s , we have seen m elements in the k^{th} component of \mathcal{A} , then the k^{th} class in \mathcal{B} has size $\langle k, m \rangle$. From the isomorphism type of the output \mathcal{B} , we can recover the set of pairs (k, m) such that the k^{th} component of the input \mathcal{A} has size m . For $1 \leq k \leq n$, if there is no such pair (k, m) , then the k^{th} component of the input is infinite.

Next, we show that $S^{n*} \leq_{tc} S^n$. Given an input structure $\mathcal{A} \in S^{n*}$, we produce an output structure $\mathcal{B} \in S^n$ that codes the number of components of each size. At stage s , we produce a sequence of numbers k_1^s, \dots, k_n^s , representing the current sizes of the components of \mathcal{A} , arranged in non-decreasing order. We give the i^{th} component of \mathcal{B} size k_i^s . \square

Lemma 11. $V^n \equiv_{tc} V^{n*}$

Proof. We first show that $V^n \leq_{tc} V^{n*}$. Recall the standard computable guessing function $d(k, s)$ such that $\liminf_s d(k, s)$ is the dimension of the k^{th} component—we are guessing at basis elements. For an input $\mathcal{A} \in V^n$, we build an output $\mathcal{B} \in V^{n*}$. At stage s , $d(k, s)$ is our guess at the dimension of the k^{th} component in \mathcal{B} . We give the k^{th} component in \mathcal{B} (this is not part of the structure) dimension $\langle k, d(k, s) \rangle$. We designate the basis elements. When the dimension of a component increases, we add new basis elements, and when the dimension decreases, we keep the first ones, and remove the last ones. From \mathcal{B} , we can recover the set of pairs (k, m) such that $m = \liminf_s d(k, s)$. For $1 \leq k \leq n$ such that $\liminf_s d(k, s) = \infty$, the k^{th} component of the input has infinite dimension.

Next, we show that $V^{n*} \leq_{tc} V^n$. For an input $\mathcal{A} \in V^{n*}$, we build an output $\mathcal{B} \in V^n$ with the same dimensions, but arranged in a non-decreasing sequence. If we guess independently the dimensions of the different components, we might always have too many of some dimension, even though the separate components drop to lower dimensions. Now, $\Delta_2^0(\mathcal{A})$ can compute the number of basis elements in each component, among the first s elements of \mathcal{A} , and arrange the numbers in a non-decreasing sequence $f(s) = d_1^s, \dots, d_n^s$. For each k , d_k^s is non-decreasing in s . If the k^{th} component of \mathcal{A} has finite dimension d , then for all sufficiently large s , $d_k^s = d$. Computably in \mathcal{A} , we guess at $f(s)$. We base our stage t construction on $f(s)$ for the first s such that our guess is new or different from the guess at the previous stage. The guesses are sensible, in that for those

s such that our current guesses at $f(s)$ seem correct, d_k^s is non-decreasing in s . Suppose at stage t , we trust our guesses at $f(0), \dots, f(s)$, and at the previous stage, we trusted our guesses at $f(0), \dots, f(s-1)$. In \mathcal{B} at the previous stage, we may have guessed $f(s)$ differently, and also guessed $f(s+1), \dots$ and, based on these guesses, we may have put into certain components of \mathcal{B} extra basis elements. We collapse the ones that are not needed for the current guess at $f(s)$. The current guess at $f(s)$ may call for new basis elements in certain components, and we add those. If for some k , there is a limiting value d for d_k^s , then in \mathcal{B} , the k^{th} component will eventually have d basis elements, and any further basis elements that we add will be collapsed. If d_k^s keeps increasing, then for each s , there is some stage t after which our guess at $f(s)$ will never change, and the k^{th} component of \mathcal{B} will eventually have d_k^s basis elements that will never be collapsed. So, the dimension is infinite. \square

4 Groups

In this section, we consider the class P^n of Abelian p -groups that are the direct sum of n groups of the form Z_{p^m} , for some m , or Z_{p^∞} (the Prüfer group). We will show that $P^n <_{tc} P^{n+1}$. We have shown that $S^n <_{tc} S^{n+1}$. Hence, it is enough to show that $P^n \equiv_{tc} S^n$.

Proposition 12. $S^n \leq_{tc} P^n$.

Proof. Given an input structure $\mathcal{A} \in S^n$, we produce an output structure $\mathcal{B} \in P^n$ such that if the k^{th} component of \mathcal{A} has size m , then the k^{th} direct summand of \mathcal{B} has type $Z_{p^{m+1}}$. \square

We have shown that $S^n \equiv_{tc} S^{n*}$. Hence, to prove $P^n \leq_{tc} S^n$, it is enough to show that $P^n \leq_{tc} S^{n*}$.

We recall some well-known facts about abelian p -groups, found in Kaplan-sky's book [10]. Recall that a subgroup H of a group G (usually Abelian) is *pure* if whenever $a \in H$ has an n^{th} root in G , there is an n^{th} root in H .

Theorem 13 ([10]). *Let G be a group and S a pure subgroup of bounded order. Then S is a direct summand of G .*

Theorem 14 ([10]). *A divisible subgroup of an abelian group is a direct summand.*

Theorem 15 ([10]). *Let G be a countable primary group with no elements of infinite height. Then G is a direct sum of cyclic groups.*

We say that decompositions of finite Abelian groups H, G such that $H \leq G$ agree with the embedding \leq if every summand of H is embedded into a summand of G under \leq .

Lemma 16. *For any finite p -groups H, G such that $H \leq G$, there are decompositions of H, G that agree with the embedding.*

Proof. Recall that if C is a pure subgroup of G and for some finite n , $p^n C = 0$ then C is a direct summand of G . Let $H = \langle h_1 \rangle \oplus \dots \oplus \langle h_n \rangle$. For all $i \leq n$, if h_i has height r and g_i is an element of G such that $p^r g_i = h_i$ then $\langle g_i \rangle$ is pure in G . Applying Theorem 13, we get $S \leq G$ such that

$$G = \langle g_1 \rangle \oplus \dots \oplus \langle g_n \rangle \oplus S \quad \& \quad H \leq \langle g_1 \rangle \oplus \dots \oplus \langle g_n \rangle$$

Taking an arbitrary decomposition of S , we have decompositions of H, G that agree with the embedding. \square

Proposition 17. $P^n \leq_{tc} S_n$.

Given $G \in P^n$, we can effectively find a chain $(G_s)_{s \in \omega}$ of finite groups such that $G = \cup_s G_s$.

Claim. We may suppose that G_s is in P^n , for all s .

Proof of Claim. In G , there are elements a_1, \dots, a_n of order p , and for each i , $1 \leq i \leq n$, there is a chain $a_{i,k}$, $k = 0, \dots$ such that $a_{i,0} = a_i$, $a_{i,k+1} = pa_{i,k}$, and G is generated by the elements $a_{i,k}$. Each finite subgroup G_s of G is contained in a subgroup H generated by $a_{1,k_1}, \dots, a_{n,k_n}$ for some choice of k_i . Clearly, $H \in P^n$. Then by Lemma 16, $G_s \in P^n$. \square

We may arrange that for each s , either $G_{s+1} = G_s$ or else G_{s+1} is generated by the elements of G_s and a single new element c such that $pc \in G_s$. For each G_s , we can find a decomposition. It would be nice if, having chosen a decomposition for G_s , we could choose a decomposition for G_{s+1} such that the decompositions agree with the identity embedding of G_s into G_{s+1} . Unfortunately, this may not be possible, as the following example illustrates.

Example 4. Suppose at stage s , we have $G_s \cong Z_{p^2} \oplus Z_{p^3}$, with generators a (for the first component) and b (for the second). For all we know at this stage, G_s may be all of G . At stage $s+1$, suppose we find G_{s+1} generated by a and an element c such that $pc = a+b$, so $G_{s+1} \cong Z_{p^2} \oplus Z_{p^4}$. There is no decomposition of G_{s+1} that agrees with the identity embedding.

Let $(G_s)_{s \in \omega}$ be a chain of finite groups as above, such that $G = \cup_s G_s$, for each s , $G_s \in P^n$, and either $G_{s+1} = G_s$ or else G_{s+1} is generated by elements of G_s and one new element c such that $pc \in G_s$. Suppose G_s has a decomposition with generators a_1, \dots, a_n , where a_i has order m_i . Let $pc = k_1 a_{i_1} + \dots + k_r a_{i_r}$, where $1 \leq k_j < m_{i_j}$. At least one k_j is not divisible by p , for otherwise, c would be in G_s . We let G_{s+1} be the decomposition of G_{s+1} generated by c and the a_i for $i \neq i_j$.

From our chosen decomposition of G_s , with direct summands $Z_{p^{m_i}}$, $m_i \geq 1$, we derive a sequence of numbers m_1, \dots, m_n . We suppose that these are arranged in non-decreasing order. For the output structure E corresponding to G , our stage s approximation has equivalence classes of sizes m_1, \dots, m_n .

While we do not label the equivalence classes, during the construction, we keep in mind the order.

Claim 1: If $u_k(G) = r$, then for all sufficiently large s , $u_k(G_s) \geq r$, and E has at least r classes of size k .

Proof of Claim 1. Suppose a generates a subgroup H of G of type $Z_{p^{k+1}}$, where a is not divisible by p in G . Once we have $a \in G_s$, the subgroup H is pure in G_s as well as in G , $u_k(G_s) \geq 1$. If $a \in G_s$, then we will see the number k on the list of sizes of sets for E . Similarly, suppose H is a subgroup of G generated by a_1, \dots, a_r , where a_i has order p^{k+1} and is not divisible by p , and the elements $p^k a_i$ witness that $u_k(G) \geq r$. Then H is pure in G_s as well as in G , and the elements $p^k a_i$ witness that $u_k(G_s) \geq r$. We will see r occurrences of the number k on our list of sizes of sets for E . \square

Claim 2: If G has r direct summands of type Z_{p^∞} , then for all k , for all sufficiently large s , $\sum_{k' \geq k} u_{k'}(G_s) \geq r$, and E has at least r infinite classes.

Proof of Claim 2. Suppose a is an element of order p and height ∞ . For each k , there is some $b \in G$ such that $p^k b = a$. If G_s contains b , then there is some $k' \geq k$ such that $u_{k'}(G_s) \geq 1$. It follows that at stage s , k' is on the list of sizes for sets in E . At each stage, we arrange the n sizes in non-decreasing order, and we build the sets in E correspondingly. If G has a direct summand of type Z_{p^∞} , then the last set in E , at least, becomes infinite. Similarly, if G has r direct summands of type Z_{p^∞} , then for all k , for all sufficiently large s , we will have at least r numbers $k' \geq k$ on the list of sizes. For each k , the last r classes in E will eventually have size at least k , so they are infinite. \square

Using the claim, we see that the numbers we get from our decompositions of the G_s give $E \in S^{n^*}$ such that there the number of classes of size k matches the number of direct summands of type $Z_{p^{k+1}}$ —this is $u_k(G)$, and the number of infinite classes matches the number of direct summands of type Z_{p^∞} .

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