

# SPECTRA OF ATOMIC THEORIES

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ABSTRACT. For a countable structure  $\mathcal{B}$ , the *spectrum* is the set of Turing degrees of isomorphic copies of  $\mathcal{B}$ . For complete elementary first order theory  $T$ , the *spectrum* is the set of Turing degrees of models of  $T$ . We answer a question from [1], showing that there is an atomic theory  $T$  whose spectrum does not match the spectrum of any structure.

## 1. INTRODUCTION

There is a substantial body of work in recursive structure theory saying when a given structure  $\mathcal{B}$  has a recursive copy, or, in case there is no recursive copy, characterizing the degrees of copies, the *spectrum of  $\mathcal{B}$* . There is also a body of work saying, for a complete elementary first order theory  $T$ , when there is a recursive model, or, in case there is no recursive model, characterizing the degrees of models, the *spectrum of  $T$* . In [9], Knight described some of the results of these kinds that were known by the early 1990's. In particular, Khisamiev [6] considered reduced Abelian  $p$ -groups of length less than  $\omega^2$ . He gave necessary and sufficient conditions for existence of a recursive copy. The conditions also say when there is an  $X$ -recursive copy, so for each reduced Abelian  $p$ -group of length less than  $\omega^2$ , we have a characterization of the spectrum.

Lempp asked what sets of degrees could serve as the spectrum of a structure. In particular, he asked whether there could be a structure with spectrum equal to the set of all non-zero degrees. Wehner [19] and Slaman [16] produced examples. There are more recent results giving other surprising structure spectra. Greenberg, Montalban, and Slaman [5] produced a structure with spectrum equal to the set of non-hyperarithmetical degrees, Csima and Kalimullin [2] produced a structure with spectrum equal to the set of hyperimmune degrees, and Diamondstone, Greenberg, and Turetsky [3] produced an structure with spectrum equal to the set of non-jump-traceable degrees.

In the early 1980's, Solovay characterized first the degrees of non-standard models of true arithmetic (TA) [17], and then the degrees of models for an arbitrary completion of  $PA$  [18]. We will say more about these results shortly. There are partial results for theories that are nicer than arithmetic in some model-theoretic way. Lerman and Schmerl [13] and Knight [8] gave sufficient conditions for an  $\aleph_0$ -categorical theory to have a recursive model. Goncharov, Harizanov, Laskowski,

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*Date:* August 13, 2012.

2010 *Mathematics Subject Classification.* Primary 03C57.

The first author was partially supported by NSF Grant Number DMS 1201338.

Both authors were locally supported by NSF Grant Number DMS 1101123.

Lempp, and McCoy [4] considered the models of a trivial strongly minimal theory. They showed that if some model is recursive, then all of the models have  $\emptyset''$ -decidable copies.

Andrews and Miller [1] began comparing theory spectra with structure spectra. They gave several examples of theory spectra which are not structure spectra, but their examples all had the feature that the theory could not have been atomic. Andrews and Miller [1, Question 1.1] asked whether for all atomic theories  $T$ , the spectrum of  $T$  is a structure spectrum. The difficulty in answering this question has two sources. First, there is no general machinery for constructing esoteric theory spectra. Andrews and Miller succeed in doing this, by coding a tree  $E \subseteq 2^{<\omega}$  into a theory, in a way that could not produce atomic theories. Second, there are only a few known negative results about spectra of structures. It is difficult to show that an upward-closed collection of degrees is not a structure spectrum. Moreover, we have mentioned above several results showing the existence of structures with unexpected spectra. In light of these examples, it should be clear why there are few theorems of the following form: If  $\mathcal{F}$  is a collection of degrees with property  $P$ , then  $\mathcal{F}$  is not the spectrum of a structure.

We answer the question of Andrews and Miller (Question 1.1) with the result below. We write  $\text{Spec}(\mathcal{A})$  for the spectrum of the structure  $\mathcal{A}$ , and we write  $\text{Spec}(T)$  for the spectrum of the theory  $T$ .

**Theorem 1.** *There is an atomic theory  $T$ , a completion of  $PA$ , such that  $\text{Spec}(T)$  is not the spectrum of any structure.*

In fact,  $\text{Spec}(T)$  will be a natural class of degrees—the degrees of non-standard models of True Arithmetic ( $TA$ ). In Section 2, we produce a completion  $T$  of  $PA$  with the desired spectrum. In Section 3, we show that this is not the spectrum of any structure. In the remainder of the introduction, we give some background on completions of  $PA$  and models of  $PA$ .

### 1.1. Completions and models of $PA$ .

**Definition.** If  $T$  is an extension of  $PA$ , then  $\text{Rep}(T)$  is the family of sets  $X \subseteq \omega$  such that for some formula  $\varphi(x)$ ,  $T \vdash \varphi(n)$  for  $n \in X$  and  $T \vdash \neg\varphi(n)$  for  $n \notin X$ .

In [15], Scott characterized the families of sets that could serve as  $\text{Rep}(T)$ , showing that they are the countable Scott sets. To show that every countable Scott set is  $\text{Rep}(T)$  for some completion  $T$  of  $PA$ , he varied the Gödel-Rosser sentence. For each  $n \geq 1$ , we can find  $\Pi_n$  sentences with the meaning “for any proof of me from axioms of  $PA$ , true  $\Sigma_{n-1}$  and  $\Pi_{n-1}$  sentences, and further sentences  $\psi_{n,0}, \dots, \psi_{n,k}$ , there is a smaller proof of my negation from these sentences”. Using this idea, we code a given set  $S$  into the  $\Sigma_n$  part of a completion of  $PA$ , leaving fixed a previously chosen  $\Sigma_{n-1}$  part. We state Scott’s technical result as follows.

**Proposition 2** (Scott). *For each  $n \geq 1$ , we have a recursive  $1-1$  function taking  $\sigma \in 2^{<\omega}$  to a  $\Pi_n$  sentence  $\psi_n(\sigma)$  such that for any completion  $T$  of  $PA$  and any set  $S \subseteq \omega$ , the following set of sentences is consistent:*

- (1) axioms of  $PA$ ,
- (2) sentences of  $T \cap (\Sigma_{n-1} \cup \Pi_{n-1})$
- (3) sentences  $\psi_n(\sigma)$ , where  $\sigma = \chi_S \upharpoonright k$  and  $k \in S$ , and sentences  $\neg\psi_n(\sigma)$ , where  $\sigma = \chi_S \upharpoonright k$  and  $k \notin S$ .

Scott [15] also considered the families of sets coded in a model  $\mathcal{M}$  of  $PA$ . For  $a \in \mathcal{M}$ , let  $X_a = \{n \in \omega : \mathcal{M} \models p_n|a\}$ . This family of sets is denoted by  $\mathcal{SS}(\mathcal{M})$ . We call this set the *Scott set* of  $\mathcal{M}$ —some people call it the *standard system*. Scott showed that for any non-standard model of  $PA$ ,  $\mathcal{SS}(\mathcal{M})$  is a Scott set. If  $\mathcal{S}$  is a Scott set that includes the elements of  $Rep(T)$ , we say that  $\mathcal{S}$  is *appropriate* for  $T$ . Clearly,  $\mathcal{S}$  is appropriate for  $T$  if it includes the fragments  $T \cap \Sigma_n$ . We note that if  $T$  is a completion of  $PA$ , and  $\mathcal{M}$  is a non-standard model of  $T$ , then  $\mathcal{SS}(\mathcal{M})$  must be appropriate for  $T$ .

Solovay [17] characterized the degrees of non-standard models of  $TA$ . Solovay’s original characterization said that the degrees of non-standard models of  $TA$  are the degrees that compute an “effective enumeration” of a Scott set containing the arithmetical sets.

**Definition.** Let  $\mathcal{S}$  be a countable family of subsets of  $\omega$ . An *enumeration* of  $\mathcal{S}$  is a binary relation  $R$  such that  $\mathcal{S}$  is the family of sets  $R^{[n]} = \{i : (n, i) \in R\}$ . If  $R^{[i]} = X$ , we call  $i$  an  *$R$ -index* for  $X$ .

**Definition.** Let  $R$  be an enumeration of a Scott set  $\mathcal{S}$ . The enumeration  $R$  is *effective* if there are recursive functions  $f, g, h$  such that

- (1)  $f(i, j)$  is an  $R$ -index for  $R_i \oplus R_j$ ,
- (2) if  $R_i$  is an infinite subtree of  $2^{<\omega}$ , then  $g(i)$  is an  $R$ -index for a path,
- (3) if  $\varphi_e^{R_i} = \chi_X$ , then  $h(e, i)$  is an  $R$ -index for  $X$ .

**Theorem 3** (Solovay). *For a set  $X$ , the following are equivalent:*

- (1)  $X$  computes a non-standard model of  $TA$ ,
- (2)  $X$  computes an effective enumeration  $R$  of a Scott set containing the arithmetical sets.

If  $\mathcal{M}$  is a non-standard model of  $PA$  with universe  $\omega$ , then the *canonical enumeration* is  $R^{\mathcal{M}} = \{(a, i) : \mathcal{M} \models p_i|a\}$ . Marker [14] showed that if  $R$  is an enumeration of a Scott set  $\mathcal{S}$ , then  $R$  computes the complete diagram of a model  $\mathcal{M}$  of  $PA$  with Scott set  $\mathcal{S}$ , and the canonical enumeration of this model is effective. This gave the following simplification of Solovay’s characterization.

**Theorem 4** (Solovay-Marker). *The degrees of non-standard models of  $TA$  are the degrees of enumerations of Scott sets containing the arithmetical sets.*

Solovay [18] characterized the degrees of models of an arbitrary completion of  $PA$ .

**Theorem 5** (Solovay [18]). *If  $T$  is a completion of  $PA$ , then for all sets  $X$  the following are equivalent:*

- (1) *There is a (nonstandard) model  $\mathcal{A}$  of  $T$  such that  $\mathcal{A} \leq_T X$ .*
- (2) *There exists an enumeration  $R \leq_T X$  of a Scott set appropriate for  $T$ , with functions  $t_n, \Delta_n^0(X)$  uniformly in  $n$ , such that  $\lim_{s \rightarrow \infty} t_n(s)$  is an  $R$ -index for  $T \cap \Sigma_n$  and for all  $s$ ,  $t_n(s)$  is an  $R$ -index for a subset of  $T_n$ .*

For an account of Solovay’s results, see [10].

**1.2. Special notation.** We will use  $\mathcal{A}$  to denote the collection of all arithmetical sets. Throughout, we let  $\mathcal{D}$  be the collection of degrees which compute enumerations of  $\mathcal{A}$ , and we let  $\mathcal{C}$  be the collection of degrees which compute an enumeration of a Scott set containing  $\mathcal{A}$ . This  $\mathcal{C}$  is precisely the degrees of non-standard models

of arithmetic, and it will be  $\text{Spec}(T)$  for the theory that we produce in the next section.

## 2. THE SPECTRUM

In this section we produce a completion  $T$  of  $PA$  such that  $\text{Spec}(T) = \mathcal{C}$ . We first use the technical result of Scott to build a completion  $T$  of  $PA$  such that  $\text{Rep}(T) = \mathcal{A}$ , and  $T \cap \Sigma_n$  is  $\Delta_n^0$ , uniformly in  $n$ , for  $n \geq 2$ . We then apply results of Solovay to show that  $\text{Spec}(T) = \mathcal{C}$ .

**Proposition 6.** *There is a completion  $T$  of  $PA$  such that*

- (1)  $\emptyset^n \leq T \cap \Sigma_{n+2}$ ,
- (2)  $T \cap \Sigma_{n+2}$  is  $\Delta_{n+2}^0$ , uniformly in  $n$ .

*Proof.* We construct  $T_n = T \cap (\Sigma_n \cup \Pi_n)$  in turn. We let  $T_0$  be the set of quantifier free consequences of  $PA$ , which is recursive. We use  $\emptyset'$  to find a completion of  $PA$ . We let  $T_1$  be the  $\Sigma_1 \cup \Pi_1$  part and we let  $T_2$  be the  $\Sigma_2 \cup \Pi_2$  part. We use Scott's technical result to code  $\emptyset'$  into  $T_3$ . For each  $k$ , if  $\sigma$  is the restriction of  $\chi_{\emptyset'}$  to  $k$ , we add the sentence  $\psi_3(\sigma)$  if  $k \in \emptyset'$ , and  $\neg\psi_3(\sigma)$  if  $k \notin \emptyset'$ . So far, we are committed to a  $\Delta_2^0$  set of sentences, consisting of the axioms of  $PA$ , sentences of  $T_2$ , and the sentences coding  $\emptyset'$ . Using  $\emptyset''$ , we can find a completion, and we let  $T_3$  be the  $\Sigma_3 \cup \Pi_3$  part. In general, given  $T_n$ , for  $n \geq 2$ , we code  $\emptyset^{n-1}$  into  $T_{n+1}$ . At this point, we are committed to a  $\Delta_n^0$  set of sentences, consisting of the axioms of  $PA$ , sentences of  $T_n$ , and the sentences  $\pm\psi_{n+1}(\sigma)$  coding  $\emptyset^{n-1}$ . Using  $\emptyset^n$ , we can find a completion, and we let  $T_{n+1}$  be the  $\Sigma_{n+1} \cup \Pi_{n+1}$  part. This completes the construction of  $T$ . □

We note that for  $n \geq 1$ ,  $TA \cap \Sigma_n$  is not  $\Delta_n^0$ —it is  $m$ -complete  $\Sigma_n^0$ . Therefore, our  $T$  is not  $TA$ , so all of the models are non-standard. We must prove the following.

**Proposition 7.**  $\text{Spec}(T) = \mathcal{C}$ .

*Proof.* Conditions (1) and (2) of Proposition 6 guarantee that  $\text{Rep}(T) = \mathcal{A}$ . Therefore,  $\text{Spec}(T) \subseteq \mathcal{C}$ ; i.e., every model computes an enumeration of a Scott set containing  $\mathcal{A}$ . We must show that if  $X$  computes an enumeration of a Scott set containing  $\mathcal{A}$ , then  $X$  computes a model of  $T$ . The lemma below will put us into a position to apply Theorem 5.

**Lemma 8.** *Suppose  $X$  computes an enumeration of a Scott set containing  $\mathcal{A}$ . Then  $X$  computes an enumeration  $R$  with a sequence of functions  $t_n$ ,  $\Delta_n^0(X)$  uniformly in  $n$ , such that  $t_n$  is constant, with value that is an  $R$ -index for  $T \cap \Sigma_n$*

*Proof of Lemma.* By the result of Solovay-Marker,  $X$  computes a non-standard model  $\mathcal{M}$  of  $TA$ . By Theorem 5, the model  $\mathcal{M}$  computes an enumeration  $R$  of a Scott set containing the arithmetical sets, with functions  $t'_n$ ,  $\Delta_n^0(X)$  uniformly in  $n$ , such that  $\lim_s t'_n(s)$  is  $TA \cap \Sigma_n$ , and for all  $s$ ,  $t'_n(s)$  is a subset of  $TA \cap \Sigma_n$ . We give ourselves  $R$ -indices for  $T \cap \Sigma_1$  and  $T \cap \Sigma_2$  and we let these be the values for our  $t_1$  and  $t_2$ . Given the constant value of  $t_n$ , we determine the constant value of  $t_{n+1}$  as follows. We note that  $T \cap \Sigma_{n+1}$  is recursive in  $\emptyset^n$  (by a uniform procedure), and  $\emptyset^n$  is recursive in  $TA \cap \Sigma_n$  (also by a uniform procedure). Using  $\Delta_{n+1}^0(X)$ , we take the limit of the  $\Delta_n^0(X)$  function  $t'_n(s)$  giving an  $R$ -index  $i$  for  $TA \cap \Sigma_n$ . From

this, we know how to compute  $\emptyset^n$  and  $T_{n+1}$ , say  $\varphi_e^{R_i} = \chi_{T \cap \Sigma_{n+1}}$ . Using  $\Delta_{n+1}^0(X)$ , or even just  $X'$ , we can find  $j$  such that  $R^{[j]} = T \cap \Sigma_{n+1}$ .  $\square$

We are now able to apply Theorem 5. If  $X$  computes an enumeration of a Scott set containing  $\mathcal{A}$ , then by the Lemma, it computes an enumeration with the extra functions  $t_n$  so that we get a model of  $T$  recursive in  $X$ . Therefore,  $\text{Spec}(T) = \mathcal{C}$ .  $\square$

### 3. NOT A STRUCTURE SPECTRUM

In this section, we show that for the theory  $T$  in the previous section,  $\text{Spec}(T)$  is not the spectrum of any structure. Here, we shall borrow heavily from forcing methods in Lachlan and Soare [11], [12]. Here is the main result from [11].

**Theorem 9** (Lachlan-Soare). *There is an enumeration  $R$  of a Scott set that includes  $\mathcal{A}$  such that  $R$  does not compute an enumeration of  $\mathcal{A}$ ; i.e.,  $\mathcal{C} \setminus \mathcal{D} \neq \emptyset$*

Here is the main result from [12].

**Theorem 10** (Lachlan-Soare). *There is an enumeration  $R$  of a family of sets that includes  $\mathcal{A}$  such that  $R$  does not compute an enumeration of a Scott set including  $\mathcal{A}$  (the degree of  $R$  is not in  $\mathcal{C}$ ).*

We first prove some facts about computation from any degree in  $\mathcal{D}$ .

**Lemma 11.** *The collection of sets that are recursively enumerable in every degree  $\mathbf{d} \in \mathcal{D}$  is equal to  $\mathcal{A}$ .*

*Proof.* We will use ideas of Lachlan and Soare. Suppose  $X \subseteq \omega$  is recursively enumerable in every enumeration of  $\mathcal{A}$ . We construct a generic enumeration  $E$  of  $\mathcal{A}$ . The forcing conditions are functions  $p$  mapping a finite initial segment of  $\omega$  to  $\mathcal{A}$ . Formally, we have  $n \in \omega$  and  $S_0, \dots, S_{n-1}$  such that  $p : n \times \omega \rightarrow 2$  and for each  $i < n$  and  $x$ ,  $p(i, x) = \chi_{S_i}(x)$ . Let  $\mathbb{E}$  be the set of these  $p$ . Our forcing language describes a structure  $(\omega, E, X)$ , where  $E \subseteq \omega \times \omega$  is the generic enumeration of  $\mathcal{A}$  being built and  $X$  is the given set. The most important statements are the ones with the meanings  $W_e^E = X$ , and  $(\exists i) E^{[i]} = S$ , for  $S \in \mathcal{A}$ . We may think of these as  $X$ -recursive infinitary formulas. Alternatively, we may add  $+$  and  $\cdot$  to the structure, so that the formulas become finitary. Our complete forcing sequence decides these statements.

Note that for any  $S \in \mathcal{A}$  and any forcing condition, some extension forces  $(\exists i) E^{[i]} = S$ . Let  $E$  be an  $\mathbb{E}$ -generic binary relation, obtained as the union of a complete forcing sequence. Suppose that  $p \Vdash W_i^E = X$ , where  $p : n \times \omega \rightarrow 2$ . If  $x \in X$ , then some  $q \supseteq p$  forces  $x \in W_e^E$ , and if  $x \notin X$ , then no  $q \supseteq p$  can force  $x \in W_e^E$ . Note that for any  $x \in \omega$  and any forcing condition  $q \supseteq p$ ,  $q \Vdash x \in W_e^E$  iff there is some finite use. We have  $x \in X$  iff there exists  $t$  and  $\sigma$  mapping a finite subset of  $(\omega - n) \times \omega$  to 2 such that  $x \in W_{e,t}^{p \cup \sigma}$ . It follows that  $X$  is c.e. relative to  $p$ . Now,  $p$  is arithmetical, so  $X$  must be as well.  $\square$

**Corollary 12.** *The collection of sets recursive in every degree  $\mathbf{d} \in \mathcal{D}$  is equal to  $\mathcal{A}$ .*

*Proof.* Any degree recursive in every  $\mathbf{d} \in \mathcal{D}$  is certainly recursively enumerable in every  $\mathbf{d} \in \mathcal{D}$ , and thus is in  $\mathcal{A}$ . Conversely, every set in  $\mathcal{A}$  is recursive (non-uniformly) in an enumeration of  $\mathcal{A}$ .  $\square$

We must show that for all structures  $\mathcal{M}$ ,  $\text{Spec}(\mathcal{M}) \neq \mathcal{C}$ . We suppose, toward a contradiction, that  $\text{Spec}(\mathcal{M}) = \mathcal{C}$ . We may suppose that  $\mathcal{M}$  has universe  $\omega$ . We let  $G_1$  and  $G_2$  be permutations of  $\omega$  that are mutually generic over  $\mathcal{M}$ . The forcing conditions are pairs  $(p_1, p_2)$ , where  $p_1$  and  $p_2$  are finite partial permutations of  $\omega$ . We think of  $p_1$  and  $p_2$  as a partial isomorphisms from the copies under construction to  $\mathcal{M}$ . We let  $(p_1, p_2) \subseteq (q_1, q_2)$  if  $p_1 \subseteq q_1$  and  $p_2 \subseteq q_2$ . We form a complete forcing sequence, and we let  $G_1$  and  $G_2$  be, respectively, the union of the first, or second components. We let  $G_1(M), G_2(M)$  be the structures such that  $G_1(M) \cong_{G_1} \mathcal{M}$ ,  $G_2(M) \cong_{G_2} \mathcal{M}$ .

**Lemma 13.**  $G_1(M) \oplus G_2(M) \in \mathcal{D}$ .

*Proof.* We may assume that  $G_1(M), G_2(M) \notin \mathcal{D}$ , since otherwise we are done. As every presentation of  $\mathcal{M}$  is in  $\mathcal{C}$ , by assumption, there must be a forcing condition  $(p_1, p_2)$  that forces, for some  $e_1$  and  $e_2$ , that  $\varphi_{e_1}^{G_1(M)}$  and  $\varphi_{e_2}^{G_2(M)}$  are total and are enumerations of Scott sets containing  $\mathcal{A}$ . Call these Scott sets  $\mathcal{S}_1$  and  $\mathcal{S}_2$ .

**Claim 14.**  $\mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{A}$

*Proof of Claim.* Suppose some finite condition  $(q_1, q_2)$ , extending  $(p_1, p_2)$ , forces that the  $i^{\text{th}}$  column of  $\varphi_{e_1}^{G_1(M)}$  is equal to the  $j^{\text{th}}$  column of  $\varphi_{e_2}^{G_2(M)}$ . For each  $x$ , there is some  $q \supseteq q_1$  such that  $(q, q_2)$  forces convergence of  $\varphi_{e_1}^{G_1(M)}(i, x)$ , and for different  $q, q' \supseteq q_1$ ,  $(q, q_2)$  and  $(q', q_2)$  cannot force convergence of  $\varphi_{e_1}^{G_1(M)}(i, x)$  to different values. Say that  $q_1$  maps  $\bar{d}$ , in the copy  $G_1(M)$  under construction, to  $\bar{c}$ , in  $\mathcal{M}$ . We can compute  $\varphi_{e_1}^{G_1(M)}(i, x)$ , given an enumeration of the existential type of  $\bar{c}$ . As we enumerate the existential type, we try mapping distinct tuples  $\bar{d}, \bar{b}$  (extending  $\bar{d}$ ) to distinct tuples  $\bar{c}, \bar{a}$  (extending  $\bar{c}$ ) and we see what would be forced if  $\bar{a}$  witnesses some formula from the existential type of  $\bar{c}$ . In this way, we find the value of  $\varphi_{e_1}^{G_1(M)}(i, x)$ . It follows from Lemma 11 that the existential type of  $\bar{c}$  is arithmetical, so column  $i$  of  $\varphi_{e_1}^{G_1(M)}$  must also be arithmetical. It follows that any set in both  $\mathcal{S}_1$  and  $\mathcal{S}_2$  is in  $\mathcal{A}$ .  $\square$

The set  $G_1(M) \oplus G_2(M)$  can compute an enumeration  $E_1$  of  $\mathcal{S}_1$  and an enumeration  $E_2$  of  $\mathcal{S}_2$ . Using these, we give an enumeration  $F$  of  $\mathcal{A}$ , where  $F$  is recursive in  $G_1(M) \oplus G_2(M)$ . We let  $F(\langle i, j \rangle, s)$  be the common value of  $E_1^{[i]}(s)$  and  $E_2^{[j]}(s)$ , if  $E_1^{[i]}(k) = E_2^{[j]}(k)$  for all  $k \leq s$ , and we let  $F(\langle i, j \rangle, s) = 0$  otherwise. The columns of  $F$  include all elements of  $\mathcal{A}$ , since all elements of  $\mathcal{A}$  are enumerated by both  $E_1$  and  $E_2$ . The only sets enumerated by both  $E_1$  and  $E_2$  are the arithmetical sets, and the sets enumerated by  $F$  either have the form  $E_1^{[i]} = E_2^{[j]} = X \in \mathcal{A}$  or are finite. Thus  $F$  is an enumeration of  $\mathcal{A}$ .  $\square$

**Lemma 15.** *There is an enumeration  $\mathcal{R}$  of a Scott set containing  $\mathcal{A}$  and there is a permutation  $G$  of  $\omega$ , generic over  $\mathcal{R}$ , such that  $\mathcal{R} \oplus G \notin \mathcal{C}$ .*

*Proof.* This is a straightforward extension of the proof of Lachlan and Soare [11] that there exists a degree in  $\mathcal{C} \setminus \mathcal{D}$ . We will present the outline of the proof and point out the one minor alteration. Let  $G_0$  be a Cohen generic, and let  $\mathcal{S}$  to be the Turing ideal generated by  $\mathcal{A}$  and  $G_0$ . Our forcing language describes a structure  $(\omega, R, G, A)$  where  $R$  is the generic enumeration of  $\mathcal{S}$  being constructed,  $G$  is the generic permutation of  $\omega$ , also being constructed, and  $A$  is a fixed enumeration of  $\mathcal{A}$ . The most important formulas say that  $\varphi_e^{\mathcal{R} \oplus G}$  is total, and that it is an enumeration of  $\mathcal{A}$ . We could use  $X$ -recursive infinitary formulas, where  $X = G_0 \oplus R$ . Alternatively, we could expand the structure being described, adding  $+$  and  $\cdot$ . Then we could use finitary formulas.

Our forcing conditions have the form  $(p, \sigma)$  where  $p : n \times \omega \rightarrow 2$ , for some  $n$ , and  $\sigma$  is a finite partial permutation of  $\omega$ . Here  $p$  is a possible assignment of  $R$ -indices to finitely many elements of  $\mathcal{S}$ , and  $\sigma$  is a possible initial segment of the permutation  $G$ . It suffices to show that no forcing condition  $(p, \sigma)$  can force  $\varphi_e^{\mathcal{R} \oplus G}$  is total and is an enumeration of  $\mathcal{A}$ . Suppose, towards a contradiction, that  $(p, \sigma)$  forces this statement. Take  $k$  such that  $(p, \sigma) \leq_T G_0 \oplus \emptyset^k$ . There must exist some  $j$  and some  $(q, \tau) \supset (p, \sigma)$  such that  $(q, \tau) \Vdash (\varphi_e^{\mathcal{R} \oplus G})^{[j]} = \emptyset^{k+3}$ . Following Lachlan and Soare, we will see that there exists a  $(q', \tau') \supset (p, \sigma)$  such that  $(q', \tau') \Vdash G_0 \leq_T (\varphi_e^{\mathcal{R} \oplus G})^{[j]}$ . This will yield a contradiction to the assumption that  $(p, \sigma) \Vdash \varphi_e^{\mathcal{R} \oplus G}$  is an enumeration of  $\mathcal{A}$ .

We identify  $2^{<\omega}$  with ways to fill in values of  $R$  on  $\text{dom}(q) \setminus \text{dom}(p)$ . Let  $U = \{\rho \mid \text{there are not two ways of determining finitely many values of } R \text{ outside of } \text{dom}(q) : \mu_1, \mu_2 \text{ and two extensions of } \tau : \tau_1, \tau_2 \text{ and a } k \text{ so that } (\varphi_e^{p \cup \rho \cup \mu_1 \cup \tau_1})^{[j]}(k) \downarrow \neq (\varphi_e^{p \cup \rho \cup \mu_2 \cup \tau_2})^{[j]}(k) \downarrow\}$ . Note that  $\rho \in U$  for any  $\rho \subset q$ . For  $\rho \in U$ , define  $\Phi(\rho, j) =$  the value to which  $(\varphi_e^{p \cup \rho \cup \mu_1 \cup \tau_1})^{[j]}(k)$  could converge for some extensions  $\mu_1$  and  $\tau_1$  as above. Let it be undefined if  $(\varphi_e^{p \cup \rho \cup \mu_1 \cup \tau_1})^{[j]}(k) \uparrow$  for every such  $\mu_1$  and  $\tau_1$ . The definition of  $U$  guarantees that this is well defined. Then let  $V = \Phi(U)$ . This  $V$  is a tree which is recursive in  $G \oplus \emptyset^{k+2}$ , but has  $\emptyset^{k+3}$  as a path. Lachlan and Soare show that any such tree is “free” on an infinite recursive set  $B = \{b_0 < b_1 < \dots\}$ . That is, for every  $h \in 2^B$ , there is an  $f \in [V]$  so that  $f \supset h$ . Lachlan and Soare then go on to show that there must be a  $q' \in [U]$  so that  $q' \leq (U \oplus G_0)''$ , so is in  $\mathcal{S}$ , and  $\Phi(q', b_k) = G_0(k)$  for each  $k \in \omega$ . As  $(p, \sigma) \Vdash \varphi_e^{\mathcal{R} \oplus G}$  is total, it follows that  $(q', \tau) \Vdash \forall k (\varphi_e^{\mathcal{R} \oplus G})^{[j]}(b_k) = G_0(k)$ , thus  $(q', \tau) \Vdash \varphi_e^{\mathcal{R} \oplus G} \geq_T G_0$ , yielding our contradiction.  $\square$

Now, there is a copy  $M$  of  $\mathcal{M}$  recursive in the enumeration  $R$  provided in Lemma 15. Thus the generic permutation  $G$  is, in fact, generic over  $M$ , and  $G$  computes a pair of permutations of  $\omega$ ,  $G_1$  and  $G_2$ , that are mutually generic over  $M$ . Consider  $G$ -recursive order-preserving bijections  $g_1 : \omega \mapsto G(\{\text{Evens}\})$  and  $g_2 : \omega \mapsto G(\{\text{Odds}\})$  and let  $G_1(x) = g_1^{-1}(G(2x))$  and  $G_2(x) = g_2^{-1}(G(2x+1))$ . Now  $G_1(M)$  and  $G_2(M)$  are mutually generic permutations of  $M$ . Therefore, by Lemma 13,  $G_1(M) \oplus G_2(M)$  computes an enumeration of  $\mathcal{A}$ . However,  $G_1(M) \oplus G_2(M) \leq_T R \oplus G$ . By Lemma 15,  $R \oplus G$  computes no enumeration of  $\mathcal{A}$ , so we have a contradiction. We conclude that it is not true that  $\text{Spec}(\mathcal{M}) = \mathcal{C}$ . This completes the proof of Theorem 1.

4. ENUMERATIONS OF  $\mathcal{A}$  THAT ARE RECURSIVELY ENUMERABLE

In this section, we prove a result about enumerations of Scott families. Although it is not used in the proof of Theorem 1 given above, this result represents an important step in our understanding of the differences between  $\mathcal{D}$  and  $\mathcal{C}$ . It indicates sharpness of the Lachlan-Soare result [11]. This result also helps to clarify the differences between enumerations that give only positive information, saying that  $x$  is in the set with index  $i$ , and enumerations that give both positive and negative information.

**Theorem 16.** *For all  $\mathbf{c} \in \mathcal{C}$ , there exists an enumeration  $R$  of  $\mathcal{A}$  that is  $\Sigma_1^c$ .*

*Proof.* We fix an enumeration  $E$  of a Scott set containing  $\mathcal{A}$ , where  $E$  has degree  $\mathbf{c}$ . We will construct an enumeration  $R$  of just  $\mathcal{A}$ , such that  $R$  is c.e. relative to  $E$ . We use a tree of guesses at  $E$ -indices for the various sets  $\emptyset^k$ . Our tree is  $\omega^{<\omega}$ , where a node  $\sigma$  represents the guess that  $E^{[\sigma(k)]} = \emptyset^k$  for all  $k < |\sigma|$ .

Fix an  $E$ -index  $m_0$  for  $\emptyset$ . For each stage  $s$ , and every  $k \in \omega$ , let  $L_1(k, s)$  be the length of agreement between  $E^{[k]} \upharpoonright s$  and  $K_s \upharpoonright s$ . At stage  $s$ , let  $m_1(s)$  be least so that  $L_1(m_1, s) > L_1(m_1, s-1)$ . This is the index of the first column of  $E$  whose length of agreement with  $\emptyset'$  has increased. Having defined  $m_1$ , define  $L_2(k, s)$  for each  $k$  to be the length of agreement between  $E^k \upharpoonright s$  and  $K_s^{E^{[m_1(s)]}} \upharpoonright s$ . Let  $m_2$  be least so that  $L_2(m_2, s) > L_2(m_2, s')$  where  $s'$  is the greatest stage  $< s$  when  $m_1(s') = m_1(s)$ . Proceed similarly to define  $m_k(s)$  for  $k < s$  and let  $\sigma_s = m_0 m_1(s) m_2(s) \dots m_{s-1}(s)$ . Note that if  $\sigma$  is a true node; that is,  $E^{[\sigma(k)]} = \emptyset^k$  for all  $k < |\sigma|$ , then  $\sigma \subseteq \sigma_s$  for infinitely many stages  $s$ . Also, note that if  $\sigma$  is not a true node, then there are only finitely many  $s$  such that  $\sigma \subseteq \sigma_s$ .

We partition  $\omega$  into infinitely many pieces  $S_\sigma$  indexed by  $\sigma \in \omega^{<\omega}$ . A node  $\sigma$  is *visited at stage  $s$*  if  $\sigma \subseteq \sigma_s$ . If  $\sigma$  is visited at stage  $s$ , then  $\sigma$  has the right to enumerate an element into each of the columns  $R^{[i]}$  for  $i \in S_\sigma$ . If  $i$  is the  $j^{\text{th}}$  number in  $S_\sigma$ , then  $\sigma$  is attempting to code  $\varphi_j^{E^{\sigma(|\sigma|-1)}}$  in  $R^{[i]}$ . It takes the least element  $x$  not yet enumerated into  $R^{[i]}$  so that  $\varphi_j^{E^{\sigma(|\sigma|-1)}}(x) \downarrow = 1$  and enumerates it into  $R^{[i]}$ .

We claim that this procedure yields an enumeration  $R$  of the family  $\mathcal{A}$ . Consider the  $j^{\text{th}}$  number in  $S_\sigma$ , and the corresponding column  $R^{[i]}$ . If  $\sigma$  is not a true guess, then  $\sigma$  is visited only finitely often, and the column  $R^{[i]}$  is finite. If, on the other hand,  $\sigma$  is a correct guess, then the image of  $\varphi_j^{E^{\sigma(|\sigma|-1)}}$  is enumerated onto the column  $R^{[i]}$ , and is arithmetical. Furthermore, every arithmetical set is  $\varphi_j^{\emptyset^k}$  for some  $j$  and  $k$ , so it is enumerated in the column corresponding the the  $j^{\text{th}}$  element of the correct  $\sigma$  of length  $k+1$ . □

Having shown in our main theorem that one particular non-standard completion of PA has the property that  $\text{Spec}(T)$  is not a structure spectrum, we conclude by asking whether a more general result is true.

**Question 17.** Let  $T$  be a completion of PA, where  $T \neq TA$ . Could  $\text{Spec}(T)$  be a structure spectrum?

We conjecture that the answer is negative.

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