
Self-full ceers and the uniform join operator

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Abstract

A computably enumerable equivalence relation (ceer) X is called self-full if whenever f is a reduction of X to X , then the range of f intersects all X -equivalence classes. It is known that the infinite self-full ceers properly contain the dark ceers, i.e. the infinite ceers which do not admit an infinite computably enumerable transversal. Unlike the collection of dark ceers, which are closed under the operation of uniform join, we answer a question from [4] by showing that there are self-full ceers X and Y so that their uniform join $X \oplus Y$ is non-self-full. We then define and examine the hereditarily self-full ceers, which are the self-full ceers X so that for any self-full Y , $X \oplus Y$ is also self-full: we show that they are closed under uniform join and that every non-universal degree in $\mathbf{Ceers}_{\mathcal{L}}$ have infinitely many incomparable hereditarily self-full strong minimal covers. In particular, every non-universal ceer is bounded by a hereditarily self-full ceer. Thus, the hereditarily self-full ceers form a properly intermediate class in between the dark ceers and the infinite self-full ceers, which is closed under \oplus .

Keywords: Computably enumerable equivalence relation, self-full equivalence relation, computable reducibility on equivalence relations

1 Introduction

We investigate the notion of self-fullness, which has turned out to be quite important in the study of computably enumerable equivalence relations (called ceers) under computable reducibility. We recall that if R, S are equivalence relations on the set of natural numbers ω , then R is computably reducible (or, simply, reducible) to S (notation: $R \leq S$) if there is a computable total function f such that

$$(\forall x, y)[x R y \Leftrightarrow f(x) S f(y)].$$

In this case, we write $f : R \rightarrow S$. This reducibility (due to Ershov [6], see also [5]) has been widely exploited in recent years as a convenient tool for measuring the computational complexity of classification problems in computable mathematics. For instance (see [8]), the isomorphism relation for various familiar classes of computable groups is Σ_1^1 -complete under \leq . Restricted to ceers, this reducibility has been used to study familiar equivalence relations from logic, such as the provable equivalence relation of sufficiently expressive formal systems (see [1] for a survey), word problems

and isomorphism problems of finitely presented groups ([12, 13]), c.e. presentations of structures (see e.g. [9, 11]).

As a reducibility, \leq gives rise in the usual way to a degree structure. Due to the importance of ceers within general equivalence relations on ω , a considerable amount of attention has been given to its substructure, called **Ceers**, consisting of the degrees of ceers. The systematic study of **Ceers**, a poset with a greatest element (usually called the universal element), was initiated by Gao and Gerdes [10]. Its algebraic structure, in particular its structure under joins and meets, was thoroughly investigated by Andrews and Sorbi [4], who have proposed a partition of the ceers into the three following classes: the finite ceers, i.e. the ceers with only finitely many equivalence classes; the light ceers, i.e. the ceers R possessing an infinite c.e. transversal (an infinite c.e. set W such that $x \not R y$ for all $x, y \in W$ with $x \neq y$), or equivalently the ceers R such that $\text{Id} \leq R$, where Id denotes the identity ceer; the dark ceers, i.e. the ceers which are neither finite nor light. These classes have been extensively investigated in relation to the existence or non-existence of joins and meets in the poset **Ceers**. For instance, no pair of incomparable degrees of dark ceers has join or meet. The classes of degrees corresponding to the classes of the above partition are first-order definable in **Ceers** in the language of posets.

Another class of ceers which has emerged in [4] consists of the self-full ceers, where a ceer R is self-full if and only if whenever a computable function f provides a reduction $f : R \rightarrow R$, then $\text{range}(f)$ intersects all R -equivalence classes. Equivalently, R is self-full if and only if $R \oplus \text{Id}_1 \not\leq R$, where Id_1 is the ceer with exactly one equivalence class. Notice that a degree containing a self-full ceer consists in fact only of self-full ceers. Let \mathcal{I} , **Dark** and **SF** respectively denote the classes of finite ceers, dark ceers and self-full ceers. It was shown in [4] that $\mathcal{I} \cup \text{Dark} \subseteq \text{SF}$ and the inclusion is proper. This follows for instance from the fact ([4, Theorem 4.10]) that every non-universal degree in **Ceers** is bounded by a self-full degree. Clearly, a self-full ceer bounding a light ceer cannot be dark. In fact it can be even proved ([4, Theorem 7.9]) that in **Ceers**/ \mathcal{I} every non-universal element has infinitely many distinct self-full strong minimal covers. In Theorem 3.10, we show the analogous result for the hereditarily self-full degrees. The partial order **Ceers**/ \mathcal{I} is defined by quotienting **Ceers** modulo $\equiv_{\mathcal{I}}$, where $R \leq_{\mathcal{I}} S$ if there is some $F \in \mathcal{I}$ so that $R \leq S \oplus F$. Recall \oplus is the operation of uniform join on equivalence relations, for which given equivalence relations R, S , we define $R \oplus S = \{(2x, 2y) : (x, y) \in R\} \cup \{(2x + 1, 2y + 1) : (x, y) \in S\}$.

The self-full ceers and their degrees play an important role in the theory of ceers. The following are few examples of the prominent role the self-full ceers have played in the theory of ceers. The existence in **Ceers**/ \mathcal{I} of infinitely many self-full strong minimal covers above any non-universal degree has been exploited in the recent paper [3] to show, among other things, that the first order theory of the degrees of light ceers is isomorphic to true first-order arithmetic. The self-full degrees form an automorphism base of the continuum many automorphisms of the poset **Ceers** [4, Corollary 11.5]. The degrees of self-full ceers are first-order definable in **Ceers** in the language of posets. They coincide exactly with the non-universal meet-irreducible degrees [4, Theorem 7.8].

The three classes of the partition of ceers (finite, light and dark) introduced in [4] are closed under the operation of uniform join. Closure under \oplus is an important issue for a class of ceers, not only as regards the investigation of the existence of infima or suprema of the degrees of the ceers in the class but also because uniform joins correspond to coproducts in the category of equivalence relations. If R, S are equivalence relations, then a morphism $\mu : R \rightarrow S$, from R to S , is a function mapping R -equivalence classes to S -equivalence classes, for which there is a computable function f such that for every x , μ maps the R -equivalence class of x to the S -equivalence class of $f(x)$. So a class of ceers is closed under \oplus if and only if the corresponding full subcategory of ceers is closed under coproducts of any pair of objects. The category-theoretic approach to numberings and equivalence

relations is due to Ershov, see in particular [7]. Since reductions correspond to monomorphisms, by [4, Lemma 1.1.] the category-theoretic jargon allows for yet another characterization of the self-full ceers, namely a ceer R is self-full if and only if every monomorphism $\mu : R \rightarrow R$ is an isomorphism.

A natural question is therefore whether or not the self-full ceers are closed under \oplus (see [4, Question 1]). We answer this question (Theorem 2.1) by showing that there are self-full ceers X, Y such that $X \oplus Y$ is not self-full. Notice that both X and Y must be infinite since $F \oplus X$ is self-full whenever F is finite and X is self-full [4, Corollary 4.3].

Motivated by the fact that the uniform join of two self-full ceers need not be self-full, in the last section of the paper, we introduce the notion of a ceer X being hereditarily self-full, i.e. $X \oplus Y$ is self-full whenever Y is self-full. Let HSF be the class of hereditarily self-full ceers. It follows from [4, Corollary 4.3] that HSF contains the finite ceers. On the other hand, we show in Corollary 3.9 that all dark ceers are hereditarily self-full. However, we show that this inclusion is proper, i.e. $\text{Dark} \subset \text{HSF}^\infty$ (where the latter class HSF^∞ consists of the infinite hereditarily self-full ceers). In Theorem 3.10, we show that every \mathcal{I} -degree has infinitely many incomparable hereditarily self-full strong minimal covers. In particular, every non-universal degree is bounded by an hereditarily self-full degree, so there are light hereditarily self-full degrees. It follows that

$$\text{Dark} \cup \mathcal{I} \subset \text{HSF} \subset \text{SF}.$$

Finally, we make two observations; first in Observation 3.16 that there is no ‘hereditarily hereditarily self-full’ ceer. That is, for every hereditarily self-full X we can find a self-full ceer Y so that $X \oplus Y$ is not hereditarily self-full. Second in Observation 3.17 that the hereditarily self-full ceers are closed under \oplus so if the Y above is hereditarily self-full (instead of just self-full) then $X \oplus Y$ is hereditarily self-full as well. Thus, the hereditarily self-full ceers can, in future analysis, often replace the self-full ceers in cases where a class which is closed under \oplus is needed.

Throughout the paper, if R is an equivalence relation on ω , and $x \in \omega$, then $[x]_R$ will denote the R -equivalence class of x . In the proofs of Theorem 2.1 and Theorem 3.15, we will be building ceers, satisfying certain requirements, using the ‘collapsing’ technique. That is, when building a ceer R in stages, we start with $R_0 = \text{Id}$. The ceer R_{s+1} will be an extension of R_s obtained by R -collapsing at $s + 1$ finitely many pairs of numbers (k, k') , where we say that we R -collapse k, k' at $s + 1$ if we merge together in R_{s+1} the R_s -equivalence classes $[k]_{R_s}$ and $[k']_{R_s}$, so that R_{s+1} is the equivalence relation generated by R_s along with the finitely many pairs of numbers which we R -collapse at $s + 1$.

2 Self-fullness is not closed under uniform join

In this section, we show that the collection of self-full ceers is not closed under \oplus .

THEOREM 2.1

There are self-full ceers X and Y so that $X \oplus Y$ is non-self-full.

PROOF. We construct a ceer Z , which will be $X \oplus Y$ for ceers X and Y . We let X refer to $Z \upharpoonright \text{Evens}$ (i.e. $x X y$ if and only if $2x Z 2y$) and Y refer to $Z \upharpoonright \text{Odds}$ (i.e. $x Y y$ if and only if $(2x+1) Z (2y+1)$). We enumerate the ceer Z in stages. At stage s , we define a ceer Z_s so that $Z = \bigcup_s Z_s$ is our desired final ceer. We build Z_{s+1} extending Z_s by the collapsing technique described at the end of Section 1. Finally, we let $X_s = Z_s \upharpoonright \text{Evens}$, and similarly $Y_s = Z_s \upharpoonright \text{Odd}$. X -collapsing k and k' at any stage means that we Z -collapse $2k$ and $2k'$. Similarly, Y -collapsing k and k' means that we Z -collapse $2k + 1$ and $2k' + 1$.

We also construct a reduction function $f : Z \rightarrow Z$. In addition to ensuring that f is a reduction and that its image omits the Z -class of 0, we have the following requirements:

$SF_{j,k}^X$: if φ_j is a reduction of X to X , then $[k]_X$ intersects the range of φ_j .
 if φ_j is a reduction of Y to Y , then $[k]_Y$ intersects the range of φ_j .

We will often say that a number z is in X if z is even and z is in Y if z is odd. We will also say that some numbers will be X -bound and other numbers may be Y -bound. Namely, when we say that a number u is X -bound, this means that if l is the greatest number such that $f^{(l)}(u)$ has been already defined then as we choose further f -images of the number, i.e. we define $f^{(m)}(u)$ for $m > l$, we will choose from X (in other words, the image will be an even number). When we say that u is Y -bound, then we will choose from Y (in other words, the image will be an odd number).

We describe a module called $\text{DiagonalizationModule}(x)$ that we will use to ensure that φ_j is not a reduction of X to X . We call this module (at some stage s) when we have an element x such that $2x$ is Y -bound and we have $\varphi_j(x) \downarrow$ and $\varphi_j(x) \not\downarrow_s x$. We may also use this module to ensure that φ_j is not a reduction of Y to Y if we have an element y such that $2y + 1$ is X -bound and we have $\varphi_j(y) \downarrow$ and $\varphi_j(y) \not\downarrow_s y$. Though we discuss the diagonalization module for X , and the strategy for $SF_{j,k}^X$, everything is symmetric for Y . \square

REMARK 2.2

When dealing with SF^X -requirements, we let $\hat{f}(u) = \frac{f(2u)}{2}$, if $f(2u) \in X$. We define $\hat{f}^{(n)}(u) = v$ if and only if $f^{(n)}(2u) = 2v$. Notice that we abuse the notation in that $\hat{f}^{(n)}$ may properly contain the n th iterate of the function \hat{f} . If $\hat{f}^{(n)}(u)$ and $\hat{f}^{(n)}(v)$ are both defined, then it is easy to see, using that $f : X \oplus Y \rightarrow X \oplus Y$ is a reduction, that $u X v$ if and only if $\hat{f}^{(n)}(u) X \hat{f}^{(n)}(v)$. Symmetrically, when dealing with SF^Y -requirements, we let $\hat{f}(u) = \frac{f(2u+1)-1}{2}$. In this case $\hat{f}^{(n)}(u) = v$ if and only if $f^{(n)}(2u + 1) = 2v + 1$.

Informal description of DiagonalizationModule(x)

Goal 1. The first goal of the module is to find some element z (which might not be x) so $2z$ is Y -bound and a stage $t > s$ so that for every n such that $\hat{f}^{(n)}(z)$ is defined, $\varphi_j(z) \not\downarrow_t \hat{f}^{(n)}(z)$. More precisely, let S be the set of n for which, by stage s , $f^{(n)}(2x)$ is determined and in X . Notice that no other $f^{(m)}(2x)$ is in X (since $2x$ is Y -bound so all later choices of $f^{(m)}(2x)$ will be odd). If $\varphi_j(x) \not\downarrow_s \hat{f}^{(n)}(x)$ for any $n \in S$, then we let $z = x$, and we move to Goal 2.

Otherwise, wait for a stage $t > s$ such that at this stage $\varphi_j(\hat{f}^{(n)}(x))$ converges for each $n \in S$. As we will argue in Lemma 2.11, either we see at this point that φ_j is not a reduction on the elements $\{\hat{f}^{(n)}(x) : n \in S\}$, or we see that there is some $m \in S, m > 0$, such that, taking $z = \hat{f}^{(m)}(x)$, we have that $\varphi_j(z) \not\downarrow_s \hat{f}^{(n)}(z)$, for every n so that $\hat{f}^{(n)}(z)$ is defined. After this we move to Goal 2.

Goal 2. We are given a z so that $2z$ is Y -bound and so that $\varphi_j(z)$ converges by stage t and is not X_t -equivalent to any $\hat{f}^{(n)}(z)$ which is already defined. We consider a new element w and wait for a stage $s' > t$ at which $\varphi_j(w)$ converges. In the meantime, we ensure that $f^{(n)}(2w)$ is in X if and only if $f^{(n)}(2z)$ is in X . The fact that w is new will ensure that the class of $2w$ does not intersect the range of f . Thus, we may later X -collapse w with z (by Z -collapsing $2w$ with $2z$) if we so wish. Once $\varphi_j(w)$ converges at s' , if $\varphi_j(w) \not\downarrow_{s'} \varphi_j(z)$, then we X -collapse w with z (i.e. we Z -collapse $2w$ with $2z$). Since $\varphi_j(z)$ is not X -equivalent to any element $\hat{f}^{(n)}(z)$ and the class of $2w$ does not intersect the image

of f , this does not cause $\varphi_j(z)$ to X -collapse with $\varphi_j(w)$. Indeed, the X -collapse of w with z entails only (as f is a reduction from Z to Z) the Z -collapse of each $f^{(n)}(2z)$ with $f^{(n)}(2w)$, i.e. the X -collapse of $\hat{f}^{(n)}(z)$ with $\hat{f}^{(n)}(w)$. Therefore, $\varphi_j(z)$ is not X -collapsed to any element as a consequence of the execution of Goal 2.

We have thus diagonalized to ensure that φ_j is not a reduction of X to X .

We now describe the strategy to satisfy $SF_{j,k}^X$. The strategy to satisfy $SF_{j,k}^Y$ is symmetric.

Informal description of the $SF_{j,k}^X$ -strategy First, suppose that $2k$ is either not bound by a higher-priority requirement or is Y -bound by a higher-priority requirement. If it is not bound by a higher-priority requirement, then we make $2k$ Y -bound. When the strategy is called at stage s , we may have already defined $f(2k), \dots, f^{(l)}(2k)$ for some l . This means that we will choose $f^{(m)}(2k)$ to be in Y (i.e. odd) for $m > l$. Next, we wait for $\varphi_j(k)$ to converge, at, say, stage s . If $\varphi_j(k) \in X_s$, then we have satisfied the requirement. Otherwise, we use $\text{DiagonalizationModule}(k)$ to ensure that φ_j is not a reduction of X to X .

Next, we consider the case where $2k$ is X -bound by a higher-priority requirement, thus we cannot use the previous strategy. We begin by choosing a new element k' , and we ensure that if we have defined $f(2k), \dots, f^{(l)}(2k)$ already, then we define $f^{(m)}(2k')$ for $m \leq l$ to be in X if and only if $f^{(m)}(2k)$ is in X , and we make $2k'$ X -bound. Next, we wait for $\varphi_j(k')$ to converge, at, say, stage s . If $\varphi_j(k')$ converges and is already X -equivalent to k' , then we X -collapse k' with k and we are done. Otherwise, $\varphi_j(k')$ converges and is not X_s -equivalent to k' . In this case, we determine that $2k'$ is Y -bound and call $\text{DiagonalizationModule}(k')$ to ensure that φ_j is not a reduction of X to X .

Construction: As anticipated at the beginning of the proof, we enumerate our desired ceer Z in stages, so that at stage s , Z_s will be a ceer, and we let X_s be $Z_s \upharpoonright \text{Evens}$ and let Y_s be $Z_s \upharpoonright \text{Odds}$.

We say that a requirement $SF_{j,k}^X$ or $SF_{j,k}^Y$ requires attention at stage s if it has not acted since it was last initialized, or if it sees some computations $\varphi_j(u)$ converge for some numbers u , and it had been waiting for these computations to converge. Requirements may determine that numbers are X -bound or Y -bound. When we write that a strategy makes a number k X -bound or Y -bound, it simultaneously makes $f^{(n)}(k)$ X -bound (or Y -bound) for every n so that $f^{(n)}(k)$ is already defined. If no active strategy is making a number X -bound or Y -bound, then we say that the number is free. A number is new at stage $s + 1$, if it is bigger than $s + 1$ and none of its X_{s-1} , Y_{s-1} or Z_{s-1} -equivalence classes contains any number so far used in the construction.

We initialize a requirement $SF_{j,k}^X$ or $SF_{j,k}^Y$ by reverting it back to the beginning of its strategy, i.e. to the initial distinction between Case 1 and Case 2 described below.

Stage 0. All requirements are initialized. Let $Z_0 = \text{Id}$.

Stage $s + 1$. Let $SF_{j,k}^X$ (or $SF_{j,k}^Y$ for which we act symmetrically, by just replacing X with Y and the even numbers with the odd numbers) be the highest-priority requirement which requires attention at stage $s + 1$. Since almost all requirements are initialized, there is such a requirement. Then, we re-initialize all requirements of lower-priority and act as follows:

– The requirement requires attention because it is initialized. We distinguish the following two cases:

Case 1. The number $2k$ is free or Y -bound. If $2k$ is currently free, then we make $2k$ Y -bound. Wait for a stage $t > s + 1$ where $\varphi_j(k)$ converges and $\varphi_j(k) \notin X_t$. We say that the strategy has entered the *Case 1-waiting outcome*.

Case 2 The number $2k$ is X -bound.

Step 0 of Case 2. Let k' be a new element. Let n be greatest so that $f^{(n)}(2k)$ is already defined. Define $f^{(m)}(2k')$ to be new elements for each $m \leq n$ ensuring that $f^{(m)}(2k')$ is in X if and only if $f^{(m)}(2k)$ is in X . Make $2k'$ X -bound. Wait for $\varphi_j(k')$ to converge. We say that the strategy has entered the Case 2 Step 0-waiting outcome.

– The requirement requires attention because it was in the Case 1-waiting outcome or in the Case 2 Step 0-waiting outcome, and now the awaited computations have converged. If it was in the Case 1-waiting outcome, then we will return to Step 1 of *DiagonalizationModule*(x) (as described below) with $x := k$. If it was in the Case 2 Step 0-waiting outcome, then it will return to Case 2 Step 1.

Step 1 of Case 2. If $\varphi_j(k') X_s k'$, then X_{s+1} -collapse k' with k . Declare the requirement satisfied. Otherwise, make $2k'$ Y -bound and go to Step 1 of *DiagonalizationModule*(x) (as described below) with $x := k'$.

– If the requirement requires attention because it was in the *DiagonalizationModule*(x) Step 1-waiting outcome, and now the awaited computation has converged, then it will return to Step 2 of *DiagonalizationModule*(x):

– If the requirement requires attention because it was in the *DiagonalizationModule*(x) Step 3(z)-waiting outcome, and now the awaited computation has converged then it will return to Step 4(z, w) of *DiagonalizationModule*(x).

DiagonalizationModule(x) (Called at $s + 1$ for a Y -bound $2x$, with $\varphi_j(x) \downarrow$ and $\varphi_j(x) \not\downarrow x$.)

Step 1. We define S the set of n so that $f^{(n)}(2x)$ is defined by stage s and is in X . Notice that since $2x$ is Y -bound, then we will not define any $f^{(m)}(2x)$ to be in X at a later stage. If $\varphi_j(x) \not\downarrow \hat{f}^{(n)}(x)$ for each $n \in S$, then we go to Step 3(x). Otherwise, wait for $\varphi_j(\hat{f}^{(n)}(x))$ to converge for every $n \in S$. We say the module has entered the *DiagonalizationModule* Step 1-waiting outcome.

Step 2. If the wait for the computations $\varphi_j(\hat{f}^{(n)}(x))$ in Step 1 to converge is over, then pick some $z = \hat{f}^{(n)}(x)$ for $n \in S$ so that $\varphi_j(z) \not\downarrow \hat{f}^{(m)}(z)$ for every m such that $f^{(m)}(2z)$ is determined before stage s and is in X . Then, go to Step 3(z). We will argue below (in the proof of Lemma 2.11) that either the requirement is satisfied (i.e. we see some n so that $\varphi_j(n) X k$, or φ_j is not a reduction of X to X) or that such a z must exist. Note that since $2z = f^{(n)}(2x)$, $2z$ is also Y -bound.

Step 3(z). Take a new element w . For every m for which $f^{(m)}(2z)$ is defined, define $f^{(m)}(2w)$ to be a new number so that $f^{(m)}(2w)$ is in X if and only if $f^{(m)}(2z)$ is in X . Make $2w$ be Y -bound (since $2z$ is Y -bound this preserves our ability to X -collapse w with z). Wait for $\varphi_j(w)$ to converge. We say the module has entered the *DiagonalizationModule* Step 3(z)-waiting outcome.

Step 4(z, w). If the wait for the computation $\varphi_j(w)$ in Step 3(z) to converge is over, then we have $\varphi_j(z)$ and $\varphi_j(w)$ both converged by stage s . If $\varphi_j(z) X_s \varphi_j(w)$, then we do nothing and declare the requirement satisfied. Otherwise, we collapse $z X_{s+1} w$ and declare the requirement satisfied.

We choose the first number v on which we have not defined $f(v)$. If v is X -bound, we take a new number x in X and define $f(v) = x$. Otherwise, we take a new number y in Y and define $f(v) = y$. If v was X -bound (or Y -bound), then we also make $f(v)$ be X -bound (or Y -bound).

Let Z_{s+1}^0 be the equivalence relation generated by Z_s along with any pairs that we have decided to Z -collapse during this stage.

Lastly, we do further collapses in order to ensure f is a reduction. That is, we let Z_{s+1} be the result of closing Z_{s+1}^0 under the implication: If $z Z_{s+1} w$ and $f(z)$ and $f(w)$ are defined, then $f(z) Z_{s+1} f(w)$.

Note that since f is only defined on finitely many values, this causes finitely much further collapse. We then stop the stage and go to stage $s + 2$.

Verification: We will adopt the following notation throughout the verification: given a parameter x chosen by an SF -strategy R , we will write $\tilde{x} = 2x$ if $R = SF_{j,k}^X$ for some j, k and $\tilde{x} = 2x + 1$ if $R = SF_{j,k}^Y$ for some j, k .

LEMMA 2.3

Each requirement is re-initialized only finitely often.

PROOF. Each requirement can act to re-initialize lower-priority requirements, without being re-initialized itself, at most finitely often. \square

In the following definition, an SF -strategy or a DiagonalizationModule module is called active for a parameter x if the module has already chosen this parameter and is waiting for a computation involving this parameter to converge.

DEFINITION 2.4

A time during the construction is either the beginning of a stage or immediately follows any Z -collapse or the assignment of any new parameter or unassignment (via initialization) of any parameters.

If s is a stage, we abuse the notation letting s also represent the time that is the beginning of the stage $s + 1$. Note that this agrees with our notation of letting Z_s be the ceer we have constructed by the end of stage s , which is the same as the beginning of stage $s + 1$.

At each time t in the construction, let D_t be the set of elements \tilde{w} which refer to parameters w for an active DiagonalizationModule which is waiting in Step 3(z).

For each time t , let E_t be the set of elements \tilde{k}' which refer to parameters k' for an active SF -strategy which is waiting in Case 2 Step 0.

For each time t , let I_t be the set of elements which are in the image of f at time t .

For a time t , we let Z_t be the ceer Z as it appears at time t . That is, if t is during stage $s + 1$, Z_t is the equivalence relation generated by Z_s and any Z -collapse so far done during stage $s + 1$. Note that if s is a stage, this definition of Z_s agrees with the previous definition.

We say that a number x is new at time t (a time during stage $s + 1$) if $x > s + 1$ and none of its X_s -, Y_s - or Z_s -equivalence classes contain any number so far used in the construction.

OBSERVATION 2.5

At each time t , the sets D_t , E_t and I_t are disjoint.

PROOF. Each of the three types of parameters: w for an active DiagonalizationModule which is in Step 3(z), k' for active SF -strategies in Case 2 Step 0, and elements $f(n)$, are chosen to be new. Thus, it is impossible that the same number is in two of these sets. \square

LEMMA 2.6

For every time t , if $x, y \in D_t \cup E_t \cup I_t$ and $x Z_t y$, then $x = f(n)$ and $y = f(m)$ and $n Z_t m$ for some n, m where $f(n)$ and $f(m)$ are defined at time t .

For any times $t \leq t'$ (i.e. t occurs before t' in the construction): if x is a number which is not new at time t and x is not Z_t -equivalent to any member of $D_t \cup E_t \cup I_t$, then x is not $Z_{t'}$ -equivalent to any member of $D_{t'} \cup E_{t'} \cup I_{t'}$.

PROOF. We prove both claims by simultaneous induction on times s . We assume the claims hold for all times $t, t' \leq s$. Let s' be the next time (i.e. exactly one action happens between s and s'). We verify that they still hold where t, t' are assumed to be times $\leq s'$. The claims clearly hold for s being the beginning of the construction (i.e. $Z_s = \text{Id}$ and no parameters have been chosen). Exactly one action takes place between times s and s' . There are three possibilities for this action:

- We may remove parameters via re-initialization or choose a new parameter which enters $D_{s'} \cup E_{s'} \cup I_{s'}$.
- We can cause Z -collapse to satisfy a requirement. This happens either between \tilde{k} and \tilde{k}' in Case 2 Step 1 of a $SF_{j,k}^X$ - or $SF_{j,k}^Y$ -strategy or between \tilde{w} and \tilde{z} in Step 4(z, w) of a DiagonalizationModule.
- We may Z -collapse $f(z)$ with $f(w)$ at the end of the stage because we have already Z -collapsed z with w .

We first consider removing or choosing new parameters. Removing parameters clearly maintain both claims since there are strictly fewer active numbers for the claims to hold for. Since all new parameters are chosen to be new numbers, their classes are singletons. Thus, both claims are maintained by assigning new parameters.

We next consider Z -collapse between \tilde{k} and \tilde{k}' in Case 2 Step 1 of a $SF_{j,k}^X$ - or $SF_{j,k}^Y$ -strategy. By the first claim of the inductive hypothesis, before this collapse \tilde{k}' was the only member of its Z_s -class which was in $D_s \cup E_s \cup I_s$. Since we make this collapse and simultaneously make the requirement satisfied (i.e. inactive), \tilde{k}' is not in $D_{s'} \cup E_{s'} \cup I_{s'}$ after this action. Thus, to the class of \tilde{k} , we have added no member of $D_{s'} \cup E_{s'} \cup I_{s'}$. We also note that the collection of numbers which are now equivalent to active numbers is a subset of the numbers that were equivalent to active numbers before the collapse. Thus, this action did not make either of the claims false.

Next, we consider Z -collapse between \tilde{w} and \tilde{z} in Step 4(z, w) of a DiagonalizationModule. Once again, \tilde{w} is no longer active after this collapse since the strategy is now satisfied and no longer active. By the first claim of the inductive hypothesis, before this collapse \tilde{w} was the only member of its Z_s -class which was in $D_s \cup E_s \cup I_s$. As in the previous case, this maintains both claims.

Finally, we consider what happens when we collapse $f(z)$ with $f(w)$ once we have already collapsed z with w . Suppose this caused $Z_{s'}$ -collapse between x and y for $x, y \in D_{s'} \cup E_{s'} \cup I_{s'}$. Note that this action does not change the values of these sets, so $x, y \in D_s \cup E_s \cup I_s$. By possibly re-naming x and y , we may assume $x Z_s f(z)$ and $y Z_s f(w)$. By inductive hypothesis, this is only possible if $x = f(n)$ and $y = f(m)$ where $n Z_s z$ and $m Z_s w$. So we see that $n Z_{s'} z Z_{s'} w Z_{s'} m$ and the first claim is preserved. Similarly, the second claim is preserved since we did not change the Z -closure of the set $D_{s'} \cup E_{s'} \cup I_{s'} = D_s \cup E_s \cup I_s$. \square

LEMMA 2.7

If x is even and y is odd, then we never Z -collapse x with y . That is, $Z = X \oplus Y$.

PROOF. We cause collapse in two cases: the first is Z -collapsing \tilde{k}' with \tilde{k} in Case 2 Step 1. In this case, since we defined f on \tilde{k}' so that $f^{(m)}(\tilde{k}')$ is in X if and only if $f^{(m)}(\tilde{k})$ is in X , and both \tilde{k} and \tilde{k}' were X -bound, or both Y -bound, when we Z -collapse \tilde{k} with \tilde{k}' (even after closing under making f a reduction), we do not cause any Z -collapses between even and odd numbers.

The second case is Z -collapsing \tilde{z} with \tilde{w} in Step 4(z, w). In this case, we have made \tilde{z} Z -collapse with \tilde{w} where we have ensured that $f^{(m)}(\tilde{w})$ is in X if and only if $f^{(m)}(\tilde{z})$ is in X and then we made both \tilde{z} and \tilde{w} be X -bound or both be Y -bound. Thus, we see that we do not cause any Z -collapse between even and odd numbers. \square

LEMMA 2.8

f is a reduction of $X \oplus Y$ to $X \oplus Y$ and $[0]_{X \oplus Y}$ is not in the range of f .

PROOF. Since whenever we have $zZ_s w$, we also cause $f(z)Z_s f(w)$, it remains to see that if $f(z)Z_s f(w)$, then we also have $zZ_s w$. This follows from the first claim of Lemma 2.6.

By the second claim in Lemma 2.6 and the fact that 0 is not new even at stage 0, 0 never becomes Z -equivalent to a number in the image of f . \square

LEMMA 2.9

Suppose x and y are mentioned by a strategy R at stage s and $x \not\sim_s y$. Suppose that $t > s$ is a stage so that the strategy R has not been re-initialized or acted at any stage between s and t . Then $x \not\sim_t y$.

PROOF. It suffices to show that no lower-priority requirement can cause a collapse which will make x Z -collapse with y . We begin by showing that any number which is not new at stage s (such as x or y) does not become $Z_{t'}$ -equivalent to parameters \tilde{k}' for a strategy of lower-priority than R in Case 2 Step 0 or parameters \tilde{w} for a DiagonalizationModule for a strategy of lower-priority than R which is waiting in Step 3(z) for any time $s < t' \leq t$. Similarly, if a is not new at stage s and is not Z_s -equivalent to a member of I_s , then a is not $Z_{t'}$ -equivalent to a member of $I_{t'}$ for any time $s < t' \leq t$.

Let t' be a time so that $s < t' \leq t$, and let a be not new at stage s . At stage s , all lower-priority requirements are re-initialized. Thus, either a is not Z_s -equivalent to a member of $D_s \cup E_s \cup I_s$, so the second claim of Lemma 2.6 ensures that it is not $Z_{t'}$ -equivalent to a member of $D_{t'} \cup E_{t'} \cup I_{t'}$, or it is Z_s -equivalent to a member of $D_s \cup E_s \cup I_s$.

If a is Z_s -equivalent to a member $u \in D_s \cup E_s$, then u is a parameter for a requirement of priority at least as high as R , which does not act or get re-initialized before stage t , so $u \in D_{t'} \cup E_{t'}$. Similarly, if $u \in I_s$, then $u \in I_{t'}$, since $I_s \subseteq I_{t'}$. The first claim of Lemma 2.6 shows that u is not $Z_{t'}$ -equivalent to parameters \tilde{k}' for a lower-priority requirement in Case 2 Step 0, or parameters \tilde{w} for a lower-priority DiagonalizationModule which is waiting in Step 3(z). And if $u \in D_s \cup E_s$, the first claim of Lemma 2.6 also shows that u is not $Z_{t'}$ -equivalent to a member of $I_{t'}$. We conclude that any number a which is not new at stage s is never equivalent to a parameter \tilde{k}' for a lower-priority requirement in Case 2 Step 0 or parameters \tilde{w} for a lower-priority DiagonalizationModule which is waiting in Step 3(z) at any t' between s and t . Further, if a is not Z_s -equivalent to a member of I_s , then a is not $Z_{t'}$ -equivalent to a member of $I_{t'}$.

We define a unique finite sequence of classes $[a_0]_{Z_s}, \dots, [a_n]_{Z_s}$ as follows. We let $a_0 = x$. Having defined a class $[a_i]_{Z_s}$, we let the class $[a_{i+1}]_{Z_s}$ be the (unique by the first claim in Lemma 2.6) class of a number u so that $f(u) \in [a_i]_{Z_s}$. If no such class exists, we simply end the finite sequence. We similarly define a sequence of classes $[b_i]_{Z_s}$ beginning with $b_0 = y$.

CLAIM 2.10 Let t' be a time so $s < t' \leq t$. Suppose that c is a number so that $f^{(k)}(c)$ is defined at time t' and is $Z_{t'}$ -equivalent to x . Then $c Z_{t'} a_i$ for some i . Similarly for y and the b_j . \square

PROOF. We distinguish between two cases. Case 1: $k > n$. Then by the first claim of Lemma 2.6 applied n times, $f^{k-n}(c) Z_{t'} a_n$. But a_n is not new at stage s and is not Z_s -equivalent to a member of I_s , so it is not $Z_{t'}$ -equivalent to a member of $I_{t'}$. This is a contradiction. Case 2: $k \leq n$. Then applying the first claim of Lemma 2.6 k times, we see that $c Z_{t'} a_k$. \square

Since no member of $[a_i]_{Z_s}$ is new at stage s for any $i \leq n$, we know that they cannot be $Z_{t'}$ -equivalent to a parameter \tilde{k}' for a lower-priority requirement in Case 2 Step 0 or parameters \tilde{w} for a lower-priority DiagonalizationModule which is waiting in Step 3(z) at any t' between s and t .

Similarly for the $[b_j]_{Z_s}$. This, along with the previous claim, shows that whenever a lower-priority requirement acts, it cannot collapse the classes of a_i 's with b_j 's. In particular, even after we ensure that f is a reduction (i.e. we collapse $f(z)$ with $f(w)$ if we have already collapsed z with w), we will not collapse x with y .

LEMMA 2.11

Each requirement is satisfied, so both X and Y are self-full.

PROOF. Let us consider a requirement $SF_{j,k}^X$. By Lemma 2.3, we can fix a stage s to be the last time this strategy will ever be re-initialized. After stage s , when it next chooses parameters k' , z and w , these choices are permanent. If we are in Case 1, then we either have $\varphi_j(k)$ diverges, or it converges and $\varphi_j(k) X k$, or we go to DiagonalizationModule(k). In the first two cases, we have satisfied the requirement because either φ_j is not total or $[k]_X$ intersects the image of φ_j . We consider the last case below. Similarly, if we are in Case 2, then we either have $\varphi_j(k')$ diverge or $\varphi_j(k') X_s k'$ at the next stage s when the requirement acts, in which case we cause $k' X k$, or we enter DiagonalizationModule(k'). In the first two cases, the requirement is satisfied because again either φ_j is not total or $[k]_X$ intersects the image of φ_j .

So, we can suppose that a DiagonalizationModule begun. If we wait in Step 1 forever, this guarantees that φ_j is not total and the requirement is satisfied. Otherwise, at some stage $s + 1$ we begin Step 2. We claim that in Step 2, either the requirement is satisfied or we must succeed in picking some z . Consider the finite set of X -classes $F = \{[\hat{f}^{(n)}(x)]_X \mid n \in S\}$. If φ_j is a reduction of X to X , then it must be injective on the classes in F . So, it either sends some class in F to a class outside of F or it is surjective on F . In the latter case, $[x]_X = [\hat{f}^{(0)}(x)]_X$ intersects the range of φ_j and the requirement is satisfied. In the former case, there is some $n \in S$ so that $\varphi_j(\hat{f}^{(n)}(x)) X \hat{f}^{(m)}(x)$ for every m . Thus, we will find the needed $z = \hat{f}^{(n)}(x)$. Thus, we begin Step 3. If we are stuck in Step 3, then φ_j is not total. If we get to Step 4(z, w) at say stage $s' + 1$, we have two cases to consider: if $\varphi_j(z) X_{s'} \varphi_j(w)$, then we do not X -collapse z with w . This guarantees that $z X_t w$ for all $t > s'$ by Lemma 2.9, therefore φ_j is not a reduction. On the other hand, if we have $\varphi_j(z) X_{s'} \varphi_j(w)$, then we X -collapse $z X_{s'+1} w$, which does not cause $\varphi_j(z) X_{s'+1} \varphi_j(w)$. By Lemma 2.9, we have $\varphi_j(z) X \varphi_j(w)$, and φ_j is not a reduction.

The proof is now complete. □

3 The hereditarily self-full ceers

We introduce the hereditarily self-full ceers and show that they properly contain the dark ceers.

DEFINITION 3.1

A ceer X is hereditarily self-full if whenever Y is self-full, $X \oplus Y$ is self-full.

Notice that by [4, Corollary 4.3] every finite ceer is hereditarily self-full.

The next definition highlights a property, which, when accompanied by self-fullness, is sufficient to guarantee hereditary self-fullness as proved in Theorem 3.3 below.

DEFINITION 3.2

We say that a ceer X is co-ceer-resistant if whenever C is a Π_1^0 -equivalence relation with infinitely many classes, then there are x, y so that $x X y$ and $x \notin C y$.

THEOREM 3.3

Every self-full co-ceer-resistant ceer is hereditarily self-full.

PROOF. Given any reduction $f : X \oplus Y \rightarrow X \oplus Y$, define $n \sim^f m$ if for every $k \in \omega$, $f^{(k)}(2n)$ is even if and only if $f^{(k)}(2m)$ is even. Note that \sim^f is a Π_1^0 -equivalence relation and X is a refinement of \sim^f (i.e. if $n \in X$ then $n \sim^f m$). We represent a \sim^f -class by a sequence in $\{X, Y\}^\omega$. Namely, we define $\tau_n^f \in \{X, Y\}^\omega$ by $\tau_n^f(k) = X$ if $f^{(k)}(2n)$ is even (or $f^{(k)}(2n) \in X$, as we will sometimes write), and $\tau_n^f(k) = Y$ if $f^{(k)}(2n)$ is odd (or $f^{(k)}(2n) \in Y$, as we will sometimes write). It is immediate to see that $n \sim^f m$ if and only if $\tau_n^f = \tau_m^f$, thus in fact we can identify the \sim^f -equivalence class of n with τ_n^f .

A sequence $\tau \in \{X, Y\}^\omega$ is eventually periodic if there exist finite sequences $\rho, \sigma \in \{X, Y\}^{<\omega}$ so that $\tau = \rho \widehat{\sigma}^\infty$, where the symbol $\widehat{}$ denotes concatenation and $\sigma^\infty = \sigma \widehat{\sigma} \widehat{\sigma} \dots \widehat{\sigma} \dots$ is the infinite string obtained by concatenating infinitely many times σ with itself. Such a string σ is a period of τ .

We say that a \sim^f -class is periodic of period k if the infinite string τ which represents \sim^f is periodic with a period of length k .

Let now X be a self-full co-ceer-resistant ceer and suppose Y is self-full. We must show that $X \oplus Y$ is self-full. Suppose that $f : X \oplus Y \rightarrow X \oplus Y$ is a reduction. We must show that the range of f intersects all the equivalence classes of $X \oplus Y$. Since $X \subseteq \sim^f$ and X is co-ceer resistant, it follows that \sim^f has only finitely many classes.

LEMMA 3.4 Each of the finitely many \sim^f -classes is represented by an eventually periodic sequence $\tau \in \{X, Y\}^\omega$. □

PROOF. Let τ be a sequence which represents a \sim^f -class, say $\tau = \tau_n^f$. If $\tau = \rho \widehat{Y}^\infty$ for some finite ρ (where, to be more precise, $Y^\infty = \langle Y \rangle^\infty$, where $\langle Y \rangle$ denotes the string of length one consisting of the sole bit Y) then τ is eventually periodic. Otherwise, let $\{k_i\}_{i \in \omega}$ be the sequence in ascending order so that $\tau(k_i) = X$ (notice that $k_0 = 0$). For every i , let $\tau_{\geq k_i}$ be the tail of τ beginning from the bit k_i of τ , i.e. $\tau_{\geq k_i}(k) = \tau(k_i + k)$. Clearly for every i there exists n_i so that $\tau_{\geq k_i} = \tau_{n_i}^f$: just take $n_i = \frac{f^{(k_i)}(2n)}{2}$. If it were $\tau_{\geq k_i} \neq \tau_{\geq k_j}$ for every i , and $j < i$, then there would be infinitely many \sim^f -equivalence classes, contrary to our previous conclusion. Therefore, there exist a least j , and for this j a least $i > j$, so that $\tau_{\geq k_i} = \tau_{\geq k_j}$. But then we see that there is a period, namely the finite string σ such that $\tau_{\geq k_j} = \sigma \widehat{\tau_{\geq k_i}}$. □

Let N_1 be a common multiple of all periods of \sim^f -classes. We replace f by $f_1 = f^{(N_1)}$, which is still a reduction of $X \oplus Y$ to $X \oplus Y$, and now all \sim^{f_1} -classes have period 1. It follows that all the τ s which represent \sim^{f_1} -classes are of the form ρX^∞ or ρY^∞ , for some ρ . Let N_2 be a number greater than the length of ρ for each \sim^{f_1} -class. Once again, we replace f_1 by $f_2 = f_1^{(N_2)}$, which ensures that each \sim^{f_2} -class is either represented by X^∞ or XY^∞ . Thus, we have at most two \sim^{f_2} -classes. Let $C_X^{f_2} = \{2n : \tau_n^{f_2} = X^\infty\}$ and $C_Y^{f_2} = \{2n : \tau_n^{f_2} = X \widehat{Y}^\infty\}$.

LEMMA 3.5

The range of $f_2 \upharpoonright C_X^{f_2}$ intersects all the $X \oplus Y$ -classes of $C_X^{f_2}$, and there are no odd numbers in $f_2^{-1}(C_X^{f_2})$.

PROOF. First of all, notice that $C_X^{f_2}, C_Y^{f_2}$ partition 2ω , they are both $X \oplus Y$ -closed, and they are decidable, being two Π_1^0 -sets that partition a decidable set. Therefore, we can define a reduction

$g : X \rightarrow X$ by:

$$g(n) = \begin{cases} \frac{f_2(2n)}{2}, & \text{if } 2n \in C_X^{f_2}, \\ n, & \text{if } 2n \in C_Y^{f_2}. \end{cases}$$

As X is self-full, it follows that the range of g must intersect all the classes of X . If it were true that $g(m) = n$ for some m, n such that $2m \in C_Y^{f_2}$ and $2n \in C_X^{f_2}$, then we would have $m = n$ by definition of g , which contradicts the fact that $C_X^{f_2}$ and $C_Y^{f_2}$ are disjoint. Therefore, it must be that the range of g restricted to $\{n : 2n \in C_X^{f_2}\}$ intersects all the X -classes of $\{n : 2n \in C_X^{f_2}\}$, and thus the range of $f_2 \upharpoonright C_X^{f_2}$ intersects all the $X \oplus Y$ -classes of $C_X^{f_2}$. Thus, there cannot be any odd $2m + 1$ such that $f_2(2m + 1) \in C_X^{f_2}$. Otherwise, let $f_2(2m + 1) = 2n \in C_X^{f_2}$, and let $2n' \in C_X^{f_2}$ be such that $f_2(2n') \equiv 2n \pmod{X \oplus Y}$; as $f_2(2m + 1) \equiv f_2(2n') \pmod{X \oplus Y}$, it would follow $2m + 1 \equiv 2n' \pmod{X \oplus Y}$, a contradiction. \square

LEMMA 3.6

For every m , either there is an odd y so that $2m + 1 \equiv f_2(y) \pmod{X \oplus Y}$ or there is an odd y so that $2m + 1 \equiv f_2^{(2)}(y) \pmod{X \oplus Y}$.

PROOF. Define the function

$$g(n) = \begin{cases} \frac{f_2(2n + 1) - 1}{2}, & \text{if } f_2(2n + 1) \text{ is odd,} \\ \frac{f_2^{(2)}(2n + 1) - 1}{2}, & \text{otherwise.} \end{cases}$$

We first observe that $g(n)$ is always an integer. If $f_2(2n + 1)$ is even, then $f_2(2n + 1)$ must be in $C_Y^{f_2}$, since $f_2^{-1}(C_X^{f_2})$ cannot contain $2n + 1$ by the previous Lemma. Thus, $f_2^{(2)}(2n + 1)$ is odd. Suppose that $n \equiv m \pmod{X \oplus Y}$. Since f_2 is a reduction of $X \oplus Y$ to $X \oplus Y$, we see that $f_2(2n + 1) \equiv f_2(2m + 1) \pmod{X \oplus Y}$, thus they have the same parity. Thus, either $g(n) = \frac{f_2(2n+1)-1}{2} \equiv \frac{f_2(2m+1)-1}{2} = g(m)$ or $g(n) = \frac{f_2^{(2)}(2n+1)-1}{2} \equiv \frac{f_2^{(2)}(2m+1)-1}{2} = g(m)$. Now, suppose that $n \not\equiv m \pmod{X \oplus Y}$. If $f_2(2n + 1)$ has the same parity as $f_2(2m + 1)$, then we can use the fact that f_2 is a reduction of $X \oplus Y$ to $X \oplus Y$ to conclude that $n \equiv m \pmod{X \oplus Y}$. If they have different parities, then we have without loss of generality: $g(n) = \frac{f_2(2n+1)-1}{2} \equiv \frac{f_2^{(2)}(2m+1)-1}{2} = g(m)$. But then we have $f_2(2n + 1) \equiv f_2^{(2)}(2m + 1) \pmod{X \oplus Y}$. But since f_2 is a reduction of $X \oplus Y$ to $X \oplus Y$, we have $2n + 1 \equiv f_2(2m + 1) \pmod{X \oplus Y}$, but the latter is even. This is a contradiction. Thus, we have shown that g is a reduction of Y to Y . Since Y is self-full, we conclude that every class of Y intersects the range of g . That is, $n \equiv g(k) \pmod{X \oplus Y}$ for some k . If $f_2(2k + 1)$ is odd, then $2m + 1 \equiv f_2(2k + 1) \pmod{X \oplus Y}$. If $f_2(2k + 1)$ is even, then $2m + 1 \equiv f_2^{(2)}(2k + 1) \pmod{X \oplus Y}$. \square

LEMMA 3.7

For every k , $[k]_{X \oplus Y}$ intersects the range of f_2 .

PROOF. If $k = 2m + 1$ then Lemma 3.6 ensures that $[k]_{X \oplus Y}$ intersects the range of f_2 .

Consider now $k = 2m$. If $2m \in C_X^{f_2}$, then we have shown in Lemma 3.5 that $[k]_{X \oplus Y}$ intersects the range of f_2 . So we suppose that $f_2(2m) = 2n + 1$. But we have shown in Lemma 3.6 that $2n + 1$ is $X \oplus Y$ -equivalent to $f_2(y)$ for an odd y or $f_2^{(2)}(y)$ for an odd y . In the former case, we have

$f_2(2m) X \oplus Y f_2(y)$, which would imply that $2m X \oplus Y y$ which is impossible since y is odd. Thus, we have $f_2(2m) X \oplus Y f_2^{(2)}(y)$, thus $2m X \oplus Y f_2(y)$. \square

This shows that the range of f_2 intersects every $X \oplus Y$ -class, so the range of f must also intersect every $X \oplus Y$ -class. Thus, $X \oplus Y$ is self-full.

LEMMA 3.8

Every dark ceer X is co-ceer resistant.

PROOF. Suppose C is a Π_1^0 -equivalence relation and has infinitely many classes. Then $\text{Id} \leq C$ because C is Π_1^0 . Indeed, if $C = W_e^c$ and W_e is the e -th c.e. set, then a reduction $f : \text{Id} \rightarrow C$ can be defined as follows: to define $f(n)$ search for the least $\langle x, s \rangle$ so that $\{f(i), x\} : i < n\} \subseteq W_{e,s}$ (such a number exists since C has infinitely many classes) and let $f(n) = x$. It follows that if X is a ceer such that $\text{Id} \not\leq X$, then there are n, m so that nXm and $n\bar{C}m$. \square

COROLLARY 3.9

Every dark ceer is hereditarily self-full.

PROOF. Every dark ceer is self-full by [4] and is co-ceer resistant by Lemma 3.8. By Theorem 3.3, dark ceers are hereditarily self-full. \square

The next theorem shows that every non-universal degree in $\mathbf{Ceers}_{\mathcal{I}}$ have infinitely many incomparable hereditarily self-full strong minimal covers. This is akin to [4, Theorem 4.10], which proves analogous result for the class of self-full ceers. Throughout the theorem and its proof, we employ the following notation. For every $k \geq 1$, we let Id_k denote a fixed ceer with exactly k classes. Note that the exact choice does not matter since any two ceers with the same finite number of classes are equivalent. For uniformity of notation, for any ceer A , we allow $A \oplus \text{Id}_0$ to represent A .

THEOREM 3.10

Let A be any non-universal ceer. Then, there are infinitely many incomparable hereditarily self-full ceers $(E_l)_{l \in \omega}$ so that for every $n, l \in \omega$ and ceer X , $A \oplus \text{Id}_n \leq E_l$ and

$$X < E_l \Rightarrow (\exists k)[X \leq A \oplus \text{Id}_k].$$

PROOF. We construct ceers E_l with the property that the function $x \mapsto 2x$ is a reduction $A \leq E_l$, satisfying the following requirements:

- CCR_j^l : if V_j is a co-c.e. equivalence relation with at least two distinct equivalence classes, then there are x, y so that $x E_l y$ and $x \not V_j y$. (Here, $V_j = W_j^c$, i.e. the complement of the c.e. set W_j .)
- $SF_i^{k,l}$: if W_i intersects infinitely many E_l -classes which do not contain an even number, then W_i intersects $[k]_X$.
- $D_j^{l,l'}$: φ_j is not a reduction of E_l to $E_{l'}$.

We first argue that any sequence of ceers $(E_l)_{l \in \omega}$ constructed with the above properties will be incomparable, will be hereditarily self-full and will have the properties that $A \oplus \text{Id}_n \leq E_l$ and $X < E_l \Rightarrow (\exists k)[X \leq A \oplus \text{Id}_k]$.

The E_l are clearly incomparable by the D -requirements. By the fact that we will make $x \mapsto 2x$ be a reduction of A to E_l , it is immediate that $A \leq E_l$ for each l . Thus, if there is some pair n, l so that $A \oplus \text{Id}_n \not\leq E_l$, then $E_l \equiv A \oplus \text{Id}_k$ for some $k \leq n$ by [4, Lemma 2.5]. Let k be minimal so that there is some E_l with $E_l \equiv A \oplus \text{Id}_k$. Then $E_l \leq E_{l'}$ for every other l' , contradicting incomparability. So, $A \oplus \text{Id}_n \leq E_l$ for every pair n, l .

Suppose that φ is a reduction witnessing $X \leq E_l$ for some ceer X . There are two cases to consider. Either φ intersects only finitely many, say, k , classes which contain no even number, in which case $X \leq A \oplus \text{Id}_k$, or by the SF -requirements applied to the c.e. image of φ , φ is onto the classes of E_l . In the latter case, $X \equiv E_l$ [4, Lemma 1.1]. This shows that $X < E_l \Rightarrow (\exists k)[X \leq A \oplus \text{Id}_k]$. Applying the same argument to any reduction φ of E_l to itself shows that φ must be onto the classes of E_l . Thus, E_l is self-full. Finally, by the CCR-requirements, E_l is also co-ceer resistant. It follows by Theorem 3.3 that the E_l are each hereditarily self-full. \square

The strategies. We now give the strategies for each requirement:

To achieve that $f(x) = 2x$ is a reduction of A to X , we guarantee that throughout the construction, we E_l -collapse a pair of distinct even numbers $2x, 2y$ if and only if $x A y$.

For the CCR_j^l -requirement, we wait until we see a pair $\langle x, y \rangle$ so that $\langle x, y \rangle$ appears in W_j . At this point, we pick a new odd z , we wait for either $\langle x, z \rangle$ or $\langle y, z \rangle$ to appear in W_j . While waiting, restrain the equivalence class of z from being E_l -collapsed to other classes due to the action of lower-priority requirements. If and when the wait is over, drop the restraint. If at that point we see $\langle x, z \rangle \in W_j$, then we E_l -collapse x and z . If we see $\langle y, z \rangle \in W_j$ but not as yet $\langle x, z \rangle \in W_j$, then we E_l -collapse y and z . Note that if V_j is an equivalence relation with at least two equivalence classes then eventually such a pair $\langle x, y \rangle$ appears in W_j . Then, for every z either $x \not\equiv z$ or $y \not\equiv z$. So whatever z we pick, eventually we see $\langle x, z \rangle \in W_j$ or $\langle y, z \rangle \in W_j$.

For the $SF_i^{k,l}$ -requirement, we wait until W_i enumerates an odd number x which is not currently E_l -equivalent to any even number or lies in any E_l -class currently restrained by a higher-priority requirement. We then E_l -collapse this x with k .

For the $D_j^{l,l'}$ -requirement: we would like to employ the natural direct diagonalization strategy, i.e. take two new numbers in E_l and choose to collapse them if and only if we do not choose to collapse their φ_j -image in $E_{l'}$. The problem is that the φ_j -images in $E_{l'}$ may be $E_{l'}$ -equivalent to even numbers (in addition to the usual problem of them being equivalent to the finitely many numbers restrained by higher-priority strategies). So, we cannot control whether or not they collapse because this is determined by A . The solution is to use the non-universality of A . In particular, as long as it appears that φ_j is giving a reduction of E_l into the classes of even numbers in $E_{l'}$ (i.e. into A), we will encode a universal ceer into E_l . This will have one of two possible outcomes: either φ_j will eventually have elements in its range which are not $E_{l'}$ -equivalent to even numbers or we will permanently witness a diagonalization. In either case, our threat to encode a universal ceer into E_l will have a finite outcome and we will not actually make E_l universal.

More explicitly, the $D_j^{l,l'}$ -strategy is as follows: we fix a universal ceer T . As we proceed, we will choose a sequence of parameters a_0, a_1, \dots which are not E_l -equivalent to any even numbers and we will ensure $a_i E_{l,s} a_k$ if and only if $i T_s k$. At every stage, we will have only chosen finitely many parameters, thus will have encoded only a finite fragment of T_s . When we see that for every pair i, k for which we have chosen parameters, $a_i E_l a_k$ if and only if $\varphi_j(a_i) E_{l'} \varphi_j(a_k)$, we will choose a new parameter a_m for the least m where we had not yet chosen a_m . In particular, we take one more step towards making $i \mapsto a_i$ be a reduction of T to $E_{l'}$. We stop this strategy if we see two numbers i, k so that $\varphi_j(a_i), \varphi_j(a_k)$ are neither $E_{l'}$ -equivalent to each other nor to an even number nor a number restrained for a higher-priority requirement. If this happens, we are ready to perform a direct diagonalization. We E_l -collapse a_i and a_k , and we restrain the pair $\varphi_j(a_i), \varphi_j(a_k)$ from any future collapse in $E_{l'}$. This strategy either ends with the direct diagonalization as described, or by φ_j being partial or by there being a pair i, k so that $a_i E_l a_k$ if and only if $\varphi_j(a_i) \not E_{l'} \varphi_j(a_k)$. In any case, φ_j will not be a reduction of E_l to $E_{l'}$.

Building the sequence $(E_l)_{l \in \omega}$ is carried out via a standard finite injury priority construction.

The construction: We build $(E_l)_{l \in \omega}$ in stages by using the collapsing technique as described at the end of Section 1.

To *initialize* a D -requirement or a CCR -requirement at a stage means to cancel its parameters, if any, and to drop its restraint, if any. Further, a D -requirement which was satisfied is no longer considered satisfied after initialization. To initialize an SF -requirement means to do nothing.

Stage 0. Let $X_0 = \text{Id}$. All requirements are initialized.

Stage $s+1$. We begin the stage by collapsing even numbers in every E_l so that $2x E_{l,s} 2y$ whenever $x A_s y$.

If $R = CCR_j^l$, we say that R is already satisfied at $s+1$ if there is $\langle w, z \rangle \in W_{j,s}$ so that $w E_{l,s} z$. If $R = SF_i^{k,l}$, we say that R is already satisfied at $s+1$ if there is $x \in W_{i,s}$ so that $x E_{l,s} k$.

A new number x at stage $s+1$ is an odd number bigger than any number so far mentioned in the construction. In particular, if x is new at stage $s+1$, then $[x]_{E_{l,s}} = \{x\}$ for each l .

Scan the requirements R in decreasing order of priority. Suppose you have scanned all $R' < R$ and distinguish the following cases:

$R = CCR_j^l$. If R is already satisfied then it drops its restraint if any, and we move to the next requirement.

If the requirement is not already satisfied at $s+1$ and it is initialized (i.e. has no chosen parameters), but we now see numbers x, y with $\langle x, y \rangle \in W_{j,s}$, then take a new odd number z which becomes the current witness of R . Restrain the equivalence class of z from being E_l -collapsed to other classes due to lower-priority requirements. After this, if we see $\langle x, z \rangle \in W_{j,s}$ or $\langle y, z \rangle \in W_{j,s}$, then drop the restraint, E_l -collapse x and z or y and z according to the case. Whether CCR_j^l chooses z or we cause E_l -collapse, we say that CCR_j^l has acted at stage $s+1$. After this action, stop the stage and initialize all lower-priority requirements. If we neither choose z nor cause E_l -collapse, then proceed to the next requirement.

$R = SF_i^{k,l}$. Suppose the requirement is not already satisfied at $s+1$, but we now see an odd number $x \in W_{i,s}$ not in the equivalence class of any number currently restrained by some higher priority R' , nor $E_{l,s}$ -equivalent to any even number. We view this x as the once-and-for-all witness of the requirement. Then, E_l -collapse x with k . We say that $SF_i^{k,l}$ has acted at stage $s+1$. After the action, stop the stage and initialize all requirements R' having lower priority than R . If no such x is found, then proceed to the next requirement.

$R = D_j^{l,l'}$. If R is currently initialized and not yet satisfied, then choose two new odd numbers a_0, a_1 which become the current witnesses of R . Restrain the classes of the two witnesses from E_l -collapses due to lower-priority requirements. After choosing a_0 and a_1 and placing restraint, stop the stage and initialize all requirements R' having lower priority than R .

If R already has witnesses, let m be the least so that a_m is not yet chosen. For $i, k < m$, E_l -collapse a_i with a_k if and only if $i T_s k$ (recall that T is our fixed universal ceer). If this causes any new collapses, then we say we have caused E_l -collapse. We distinguish the following three cases:

Case 1: there are $i, k < m$ so that $\varphi_j(a_i), \varphi_j(a_k)$ both converge by stage s to numbers which are not $E_{l',s}$ -equivalent to an even number nor a number restrained by higher-priority requirements and $\varphi_j(a_i) E_{l',s} \varphi_j(a_k)$. We E_l -collapse a_i with a_k and place a permanent restraint to not allow lower-priority requirements to cause $\varphi_j(a_i)$ to $E_{l'}$ collapse with $\varphi_j(a_k)$. We let $\varphi_j(a_i)$ and $\varphi_j(a_k)$ permanently

be parameters c and d of the requirement. We now say that the requirement is satisfied (this is permanent unless the requirement is re-initialized). After this action, stop the stage and initialize all lower-priority requirements.

Case 2: not Case 1 and $\varphi_j(a_i)$ converges by stage s for every $i < m$ and for each $i, k < m$, $a_i E_{l,s} a_k$ if and only if $\varphi_j(a_i) E_{l,s} \varphi_j(a_k)$. In this case, we define a_m to be a new odd number. After this action, stop the stage and initialize all lower-priority requirements.

Case 3: not Case 1 or Case 2: if we have caused E_l -collapse, then stop the stage and initialize all lower-priority requirements. Otherwise, move to the next requirement.

Notice that eventually we are sure to move on to stage $s + 2$ since the action of any initialized D -requirement stops the stage, and there are co-finitely many initialized requirements at any stage.

The verification: We split the verification into the following lemmata. Notice that at each stage, exactly one requirement acts.

Let us say that a witness for a requirement R is E_l -active if it is a parameter z for a CCR_j^l -requirement or is an a_i parameter for a $D_j^{l,l'}$ -requirement or is a c or d parameter for a $D_j^{l,l'}$ -requirement. That is, it is a parameter for which some requirement restrains E_l -collapse.

LEMMA 3.11

At every stage s , if x and y are E_l -active witnesses for different requirements, then $x \not E_l y$.

At every stage s , if x, y are E_l -active witnesses and $x E_{l,s} y$, then $x = a_i$ and $y = a_j$ for a D -requirement and $i T_s j$.

At every stage s , if x is E_l -active, then x is not $E_{l,s}$ -equivalent to any even number.

PROOF. We prove all three claims by simultaneous induction on s . The lemma is certainly true at stage 0, as there are no E_l -active witnesses. If the claim is true at s , then at $s + 1$ at most one R can effect E_l . Either $R E_{l,s+1}$ -collapses the class of its witness with some other class, or (if R is a D -requirement) $E_{l,s+1}$ -collapses classes of its witnesses or R appoints new E_l -active witnesses.

If R is a CCR -requirement, then R 's witness z ceases to be active. So, it has collapsed $[z]_{E_{l,s}}$ to some other class $[w]_{E_{l,s}}$. By inductive hypothesis, the only member of $[z]_{E_{l,s}}$ which was either E_l -active or even was z itself. Since z is no longer active at stage $s + 1$, we have added neither an E_l -active number nor an even number to $[w]_{E_{l,s+1}}$.

If R is a D -requirement, it collapses a_i with a_j only if $i T_s j$. Since neither a_i nor a_j were $E_{l,s}$ -equivalent to another requirement's E_l -active witness or an even number by inductive hypothesis, the combined class maintains this property.

If R is an $SF_i^{k,l}$ -requirement, then it collapses its once-and-for-all witness x to k . This x is not $E_{l,s}$ -equivalent to any $E_{l,s+1}$ -active number since it is not $E_{l,s}$ -equivalent to a number restrained by a higher-priority requirement (and this action initializes all lower-priority requirements) nor an even number, and it initializes all lower-priority requirements. Thus, it has added neither an even number nor an E_l -active number to $[k]_{E_{l,s+1}}$.

Finally, if R appoints new E_l -active witnesses, these witnesses are either new, so their classes are currently disjoint singletons, or are the c and d for a D -requirement, which are appointed only if they are in distinct E_l -classes from every other E_l -active or even number. \square

LEMMA 3.12

The map $i \mapsto 2i$ is a reduction witnessing $A \leq E_l$ for each l .

PROOF. Since no active parameter is ever equivalent to an even number by Lemma 3.11, we never cause collapse of two even numbers aside from the beginning of each stage. At that point, we collapse

$2x E_{l,s} 2y$ if and only if $x A_s y$. Since we never cause any other collapse among even numbers, we have $2x E_l 2y$ if and only if $x A y$. \square

LEMMA 3.13

Every requirement is initialized finitely often, eventually stops acting and sets up only a finite restraint.

PROOF. The proof is by induction on the priority ranking of the requirement R , since every requirement acts only finitely often after each initialization. This is clear for each requirement except D -requirements. Clearly, they can act via Case 1 at most once after last initialization. Suppose towards a contradiction that it acts infinitely often via Case 2 after last initialization. Then $\psi : i \mapsto a_i$ is a reduction of T to E_l . Further, the range of ψ is contained in the classes which contain even numbers and the finitely many (by inductive hypothesis) classes restrained for higher priority strategies. Thus, using the fact that E_l restricted to the even numbers is equivalent to A by Lemma 3.12, we can alter ψ to give a reduction of T to $A \oplus \text{Id}_k$ for some k . But since A is assumed to be non-universal, the universal degree is uniform join-irreducible [2, Proposition 2.6], $A \oplus \text{Id}_k$ is non-universal. This is a contradiction. Finally, since the requirement acts only finitely often in Case 2, it defines only finitely many parameters a_i . Thus, it can only cause E_l -collapse finitely often and can only initialize lower-priority requirements in Case 3 finitely often. So, D -requirements also act finitely often, place finite restraint and initialize lower-priority requirements finitely often after their last initialization. \square

LEMMA 3.14

Every requirement is eventually satisfied.

PROOF. Notice that CCR - or SF -requirements can never be injured after they have acted. If they collapse, then the collapse permanently satisfies the requirement. Suppose that every $R' < R$ is eventually satisfied. \square

We now distinguish the various possibilities for R .

$R = SF_i^{k,l}$. We must only worry if W_i intersects infinitely many E_l -classes not containing even numbers. In this case, eventually some odd number x appears in W_i not as yet E_l -equivalent to any even number and avoiding the finite restraint imposed by the higher-priority requirements. So R acts by picking such an x (the once-and-for-all witness of the requirement), and x is E_l -collapsed to k , which makes R permanently satisfied.

$R = CCR_j^l$. By the fact that lower-priority SF -requirements will choose their once-and-for-all witnesses avoiding higher-priority restraints, and thus not interfering with CCR_j^l , the only possible injury to CCR_j^l can be made, thanks to Lemma 3.11, by higher-priority requirements. By Lemma 3.13 let s_0 be the least stage such that these higher-priority requirements do not act at any $s \geq s_0$. If V_j is an equivalence relation with at least two distinct equivalence classes, then eventually we either see at some point that R is already satisfied without having acted (and so the requirement is permanently satisfied) or we see at some point a pair $\langle x, y \rangle$ to appear in W_j . At that point, we pick the final witness z , a new odd number. At some later stage, we collapse z with x or y , which makes R permanently satisfied.

$R = D_j^{l,l'}$ By the fact that lower-priority SF -requirements will choose their once-and-for-all witnesses avoiding higher-priority restraint, and thus not interfering with D_i , the only possible injury

to D_i can be made, thanks to Lemma 3.11, by higher-priority requirements. By Lemma 3.13 let s_0 be the least stage such that these higher-priority requirements do not act at any $s \geq s_0$. Requirement $D_j^{l,l'}$ is never re-initialized after s_0 . After stage s_0 , we have several possible outcomes for $D_j^{l,l'}$. We have already shown that it cannot take Case 2 infinitely often. If it ever takes Case 1, then we have two numbers a_i, a_k so that $a_i E_l a_k$ and $\varphi_j(a_i) \not E_{l'} \varphi_j(a_k)$. The remaining possibility is that it takes Case 3 co-finitely often. In this case, we see that either φ_j is not total or is not a reduction (either case is witnessed on the finite set $\{a_i \mid i < m\}$).

The proof of the theorem is now complete.

Note the following corollary which follows from letting A be Id:

COROLLARY 3.15

There are light ceers which are hereditarily self-full.

We conclude by showing that the property of being hereditarily self-full is itself hereditary in one sense and not in another.

OBSERVATION 3.16

If Z is hereditarily self-full, then there is a self-full ceer X so that $Z \oplus X$ is not hereditarily self-full.

PROOF. Let Z be hereditarily self-full, and let X and Y be self-full so that $X \oplus Y$ is non-self-full. Then $Z \oplus X$ is not hereditarily self-full because $Z \oplus X \oplus Y$ is non-self-full. \square

OBSERVATION 3.17

If X is hereditarily self-full and Y is hereditarily self-full, then $X \oplus Y$ is hereditarily self-full.

PROOF. Let E be any self-full ceer. Then since Y is hereditarily self-full, we see that $Y \oplus E$ is self-full. Thus, since X is hereditarily self-full, we see that $X \oplus (Y \oplus E)$ is self-full. We conclude that whenever E is self-full, $(X \oplus Y) \oplus E$ is self-full, showing that $X \oplus Y$ is hereditarily self-full. \square

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