

# DEFINABLE CLOSURE IN RANDOMIZATIONS

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ABSTRACT. The randomization of a complete first order theory  $T$  is the complete continuous theory  $T^R$  with two sorts, a sort for random elements of models of  $T$ , and a sort for events in an underlying probability space. We give necessary and sufficient conditions for an element to be definable over a set of parameters in a model of  $T^R$ .

## 1. INTRODUCTION

A randomization of a first order structure  $\mathcal{M}$ , as introduced by Keisler [Kei1] and formalized as a metric structure by Ben Yaacov and Keisler [BK], is a continuous structure  $\mathcal{N}$  with two sorts, a sort for random elements of  $\mathcal{M}$ , and a sort for events in an underlying atomless probability space. Given a complete first order theory  $T$ , the theory  $T^R$  of randomizations of models of  $T$  forms a complete theory in continuous logic, which is called the randomization of  $T$ . In a model  $\mathcal{N}$  of  $T^R$ , for each  $n$ -tuple  $\vec{a}$  of random elements and each first order formula  $\varphi(\vec{v})$ , the set of points in the underlying probability space where  $\varphi(\vec{a})$  is true is an event denoted by  $\llbracket \varphi(\vec{a}) \rrbracket$ .

In a first order structure  $\mathcal{M}$ , an element  $b$  is *definable over* a set  $A$  of elements of  $\mathcal{M}$  (called parameters) if there is a tuple  $\vec{a}$  in  $A$  and a formula  $\varphi(u, \vec{a})$  such that

$$\mathcal{M} \models (\forall u)(\varphi(u, \vec{a}) \leftrightarrow u = b).$$

In a general metric structure  $\mathcal{N}$ , an element  $b$  is said to be *definable over* a set of parameters  $A$  if there is a sequence of tuples  $\vec{a}_n$  in  $A$  and continuous formulas  $\Phi_n(x, \vec{a}_n)$  whose truth values converge uniformly to the distance from  $x$  to  $b$ . In this paper we give necessary and sufficient conditions for definability in a model of the randomization theory  $T^R$ . These conditions can be stated in terms of sequences of first order formulas. The results in this paper will be applied in a forthcoming paper about independence relations in randomizations.

In Theorem 3.1.2, we show that an event  $E$  is definable over a set  $A$  of parameters if and only if it is the limit of a sequence of events of the form  $\llbracket \varphi_n(\vec{a}_n) \rrbracket$ , where each  $\varphi_n$  is a first order formula and each  $\vec{a}_n$  is a tuple from  $A$ .

In Theorem 3.3.6, we show that a random element  $b$  is definable over a set  $A$  of parameters if and only if  $b$  is the limit of a sequence of random elements

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$b_n$  such that for each  $n$ ,

$$\llbracket (\forall u)(\varphi_n(u, \vec{a}_n) \leftrightarrow u = b_n) \rrbracket$$

has probability one for some first order formula  $\varphi_n(u, \vec{v})$  and a tuple  $\vec{a}_n$  from  $A$ . In Section 4 we give some consequences in the special case that the underlying first order theory  $T$  is  $\aleph_0$ -categorical.

Continuous model theory in its current form is developed in the papers [BBHU] and [BU]. The papers [Go1], [Go2], [Go3] deal with definability questions in metric structures. Randomizations of models are treated in [AK], [Be], [BK], [EG], [GL], [Ke1], and [Ke2].

## 2. PRELIMINARIES

We refer to [BBHU] and [BU] for background in continuous model theory, and follow the notation of [BK]. We assume familiarity with the basic notions about continuous model theory as developed in [BBHU], including the notions of a theory, structure, pre-structure, model of a theory, elementary extension, isomorphism, and  $\kappa$ -saturated structure. In particular, the universe of a pre-structure is a pseudo-metric space, the universe of a structure is a complete metric space, and every pre-structure has a unique completion. In continuous logic, formulas have truth values in the unit interval  $[0, 1]$  with 0 meaning true, the connectives are continuous functions from  $[0, 1]^n$  into  $[0, 1]$ , and the quantifiers are sup and inf. A *tuple* is a finite sequence, and  $A^{<\mathbb{N}}$  is the set of all tuples of elements of  $A$ .

**2.1. The theory  $T^R$ .** We assume throughout that  $L$  is a finite or countable first order signature, and that  $T$  is a complete theory for  $L$  whose models have at least two elements.

The *randomization signature*  $L^R$  is the two-sorted continuous signature with sorts  $\mathbb{K}$  (for random elements) and  $\mathbb{B}$  (for events), an  $n$ -ary function symbol  $\llbracket \varphi(\cdot) \rrbracket$  of sort  $\mathbb{K}^n \rightarrow \mathbb{B}$  for each first order formula  $\varphi$  of  $L$  with  $n$  free variables, a  $[0, 1]$ -valued unary predicate symbol  $\mu$  of sort  $\mathbb{B}$  for probability, and the Boolean operations  $\top, \perp, \sqcap, \sqcup, \neg$  of sort  $\mathbb{B}$ . The signature  $L^R$  also has distance predicates  $d_{\mathbb{B}}$  of sort  $\mathbb{B}$  and  $d_{\mathbb{K}}$  of sort  $\mathbb{K}$ . In  $L^R$ , we use  $B, C, \dots$  for variables or parameters of sort  $\mathbb{B}$ .  $B \dot{=} C$  means  $d_{\mathbb{B}}(B, C) = 0$ , and  $B \sqsubseteq C$  means  $B \dot{=} B \sqcap C$ .

A pre-structure for  $T^R$  will be a pair  $\mathcal{P} = (\mathcal{K}, \mathcal{B})$  where  $\mathcal{K}$  is the part of sort  $\mathbb{K}$  and  $\mathcal{B}$  is the part of sort  $\mathbb{B}$ . The *reduction* of  $\mathcal{P}$  is the pre-structure  $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$  obtained from  $\mathcal{P}$  by identifying elements at distance zero, and the associated mapping from  $\mathcal{P}$  onto  $\mathcal{N}$  is called the *reduction map*. The *completion* of  $\mathcal{P}$  is the structure obtained by completing the metrics in the reduction of  $\mathcal{P}$ . A pre-structure  $\mathcal{P}$  is called *pre-complete* if the reduction of  $\mathcal{P}$  is already the completion of  $\mathcal{P}$ .

In [BK], the randomization theory  $T^R$  is defined by listing a set of axioms. We will not repeat these axioms here, because it is simpler to give the following model-theoretic characterization of  $T^R$ .

**Definition 2.1.1.** Given a model  $\mathcal{M}$  of  $T$ , a *nice randomization* of  $\mathcal{M}$  is a pre-complete structure  $(\mathcal{K}, \mathcal{B})$  for  $L^R$  equipped with an atomless probability space  $(\Omega, \mathcal{B}, \mu)$  such that:

- (1)  $\mathcal{B}$  is a  $\sigma$ -algebra with  $\top, \perp, \sqcap, \sqcup, \neg$  interpreted by  $\Omega, \emptyset, \cap, \cup, \setminus$ .
- (2)  $\mathcal{K}$  is a set of functions  $a: \Omega \rightarrow M$ .
- (3) For each formula  $\psi(\vec{x})$  of  $L$  and tuple  $\vec{a}$  in  $\mathcal{K}$ , we have

$$\llbracket \psi(\vec{a}) \rrbracket = \{\omega \in \Omega : \mathcal{M} \models \psi(\vec{a}(\omega))\} \in \mathcal{B}.$$

- (4)  $\mathcal{B}$  is equal to the set of all events  $\llbracket \psi(\vec{a}) \rrbracket$  where  $\psi(\vec{v})$  is a formula of  $L$  and  $\vec{a}$  is a tuple in  $\mathcal{K}$ .
- (5) For each formula  $\theta(u, \vec{v})$  of  $L$  and tuple  $\vec{b}$  in  $\mathcal{K}$ , there exists  $a \in \mathcal{K}$  such that

$$\llbracket \theta(a, \vec{b}) \rrbracket = \llbracket (\exists u \theta)(\vec{b}) \rrbracket.$$

- (6) On  $\mathcal{K}$ , the distance predicate  $d_{\mathbb{K}}$  defines the pseudo-metric

$$d_{\mathbb{K}}(a, b) = \mu[a \neq b].$$

- (7) On  $\mathcal{B}$ , the distance predicate  $d_{\mathbb{B}}$  defines the pseudo-metric

$$d_{\mathbb{B}}(B, C) = \mu(B \Delta C).$$

**Definition 2.1.2.** For each first order theory  $T$ , the *randomization theory*  $T^R$  is the set of sentences that are true in all nice randomizations of models of  $T$ .

It follows that for each first order sentence  $\varphi$ , if  $T \models \varphi$  then  $T^R \models \llbracket \varphi \rrbracket \doteq \top$ . The following basic facts are from [BK], Theorem 2.1 and Proposition 2.2, Example 3.4 (ii), Proposition 2.7, and Theorem 2.9.

**Fact 2.1.3.** For every complete first order theory  $T$ , the randomization theory  $T^R$  is complete.

**Fact 2.1.4.** Every model  $\mathcal{M}$  of  $T$  has nice randomizations.

**Fact 2.1.5.** (Fullness) Every pre-complete model  $\mathcal{P} = (\mathcal{K}, \mathcal{B})$  of  $T^R$  has perfect witnesses, i.e.,

- (1) For each first order formula  $\theta(u, \vec{v})$  and each  $\vec{b}$  in  $\mathcal{K}^n$  there exists  $a \in \mathcal{K}$  such that

$$\llbracket \theta(a, \vec{b}) \rrbracket \doteq \llbracket (\exists u \theta)(\vec{b}) \rrbracket;$$

- (2) For each  $B \in \mathcal{B}$  there exist  $a, b \in \mathcal{K}$  such that  $B \doteq \llbracket a = b \rrbracket$ .

**Corollary 2.1.6.** Every model  $\mathcal{N}$  of  $T^R$  has a pair of elements  $c, d$  such that  $\llbracket c \neq d \rrbracket = \top$ .

*Proof.* Every model of  $T$  has at least two elements, so  $T \models (\exists u)(\exists v)u \neq v$ . The result follows by applying Fullness twice.  $\square$

**Fact 2.1.7.** (Strong quantifier elimination) Every formula  $\Phi$  in the continuous language  $L^R$  is  $T^R$ -equivalent to a formula with the same free variables and no quantifiers of sort  $\mathbb{K}$  or  $\mathbb{B}$ .

**Lemma 2.1.8.** *Let  $\mathcal{P} = (\mathcal{K}, \mathcal{B})$  be a pre-complete model of  $T^R$  and let  $a, b \in \mathcal{K}$  and  $B \in \mathcal{B}$ . Then there is an element  $c \in \mathcal{K}$  that agrees with  $a$  on  $B$  and agrees with  $b$  on  $\neg B$ , that is,  $B \sqsubseteq \llbracket c = a \rrbracket$  and  $(\neg B) \sqsubseteq \llbracket c = b \rrbracket$ .*

**Definition 2.1.9.** In Lemma 2.1.8, we will call  $c$  a *characteristic function* of  $B$  with respect to  $a, b$ .

Note that the distance between any two characteristic functions of an event  $B$  with respect to elements  $a, b$  is zero. In particular, in a model of  $T^R$ , the characteristic function is unique.

*Proof of Lemma 2.1.8.* By Fact 2.1.5 (2), there exist  $d, e \in \mathcal{K}$  such that  $B \doteq \llbracket d = e \rrbracket$ . The first order sentence

$$(\forall u)(\forall v)(\forall x)(\forall y)(\exists z)[(x = y \rightarrow z = u) \wedge (x \neq y \rightarrow z = v)]$$

is logically valid, so we must have

$$\llbracket (\exists z)[(d = e \rightarrow z = a) \wedge (d \neq e \rightarrow z = b)] \rrbracket \doteq \top.$$

By Fact 2.1.5 (1) there exists  $c \in \mathcal{K}$  such that

$$\llbracket d = e \rightarrow c = a \rrbracket \doteq \top, \quad \llbracket d \neq e \rightarrow c = b \rrbracket \doteq \top,$$

so  $\llbracket d = e \rrbracket \sqsubseteq \llbracket c = a \rrbracket$  and  $\llbracket d \neq e \rrbracket \sqsubseteq \llbracket c = b \rrbracket$ .  $\square$

We will need the following result, which is a consequence of Theorem 3.11 of [Be]. Since the setting in [Be] is quite different from the present paper, we give a direct proof here.

**Proposition 2.1.10.** *Every model of  $T^R$  is isomorphic to the reduction of a nice randomization of a model of  $T$ .*

*Proof.* Let  $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$  be a model of  $T^R$  of cardinality  $\kappa$ . Let  $\Omega$  be the Stone space of the Boolean algebra  $\widehat{\mathcal{B}} = (\widehat{\mathcal{B}}, \top, \perp, \sqcap, \sqcup, \neg)$ . Thus  $\Omega$  is a compact topological space, the points of  $\Omega$  are ultrafilters, we may identify  $\widehat{\mathcal{B}}$  with the Boolean algebra of clopen sets of  $\Omega$ , and  $\mu^{\mathcal{N}}$  is a finitely additive probability measure on  $\widehat{\mathcal{B}}$ .

We next show that  $\mu$  is  $\sigma$ -additive on  $\widehat{\mathcal{B}}$ . To do this, we assume that  $A_0 \supseteq A_1 \supseteq \dots$  in  $\widehat{\mathcal{B}}$  and  $C = \bigcap_n A_n \in \widehat{\mathcal{B}}$ , and prove that  $\mu(C) = \lim_{n \rightarrow \infty} \mu(A_n)$ . Indeed, the family  $\{C \cup (\Omega \setminus A_n) : n \in \mathbb{N}\}$  is an open covering of  $\Omega$ , so by the topological compactness of  $\Omega$ , we have  $\Omega = \bigcup_{k=0}^n (C \cup (\Omega \setminus A_k))$  for some  $n \in \mathbb{N}$ . Then  $C = A_n$ , so  $\mu(C) = \mu(A_n) = \lim_{n \rightarrow \infty} \mu(A_n)$ .

By the Caratheodory theorem, there is a complete probability space  $(\Omega, \mathcal{B}, \mu)$  such that  $\mathcal{B} \supseteq \widehat{\mathcal{B}}$ ,  $\mu$  agrees with  $\mu^{\mathcal{N}}$  on  $\widehat{\mathcal{B}}$ , and for each  $B \in \mathcal{B}$  and  $m > 0$  there is a countable sequence  $A_{m0} \subseteq A_{m1} \subseteq \dots$  in  $\widehat{\mathcal{B}}$  such that

$$(2.1) \quad B \subseteq \bigcup_n A_{mn} \text{ and } \mu \left( \bigcup_n A_{mn} \right) \leq \mu(B) + 1/m.$$

Note that since the probability space  $(\Omega, \mathcal{B}, \mu)$  is complete, every subset of  $\Omega$  that contains a set in  $\mathcal{B}$  of measure one also belongs to  $\mathcal{B}$  and has measure one.

We claim that for each  $\mathbf{B} \in \mathcal{B}$  there is a unique event  $f(\mathbf{B}) \in \widehat{\mathcal{B}}$  such that  $\mu(f(\mathbf{B})\Delta\mathbf{B}) = 0$ . The uniqueness of  $f(\mathbf{B})$  follows from the fact that the distance function  $d_{\mathbb{B}}(\mathbf{C}, \mathbf{D}) = \mu(\mathbf{C}\Delta\mathbf{D})$  is a metric on  $\widehat{\mathcal{B}}$ . To show the existence of  $f(\mathbf{B})$ , for each  $m > 0$  let  $\mathbf{A}_{m0} \subseteq \mathbf{A}_{m1} \subseteq \dots$  be as in (2.1). Note that  $(\mathbf{A}_{m0}, \mathbf{A}_{m1}, \dots)$  is a Cauchy sequence of events in the model  $\mathcal{N}$ , so there is an event  $\mathbf{C}_m \in \widehat{\mathcal{B}}$  such that  $\mathbf{C}_m = \lim_{n \rightarrow \infty} \mathbf{A}_{mn}$ . Hence  $\lim_{n \rightarrow \infty} \mu(\mathbf{A}_{mn}\Delta\mathbf{C}_m) = 0$ , so  $\mu((\bigcup_n \mathbf{A}_{mn})\Delta\mathbf{C}_m) = 0$ . Then  $(\mathbf{C}_1, \mathbf{C}_2, \dots)$  is a Cauchy sequence, so there is an event  $f(\mathbf{B}) = \lim_{m \rightarrow \infty} \mathbf{C}_m$  in  $\widehat{\mathcal{B}}$  with  $\mu(f(\mathbf{B})\Delta\mathbf{B}) = 0$ .

We make some observations about the mapping  $f: \mathcal{B} \rightarrow \widehat{\mathcal{B}}$ . If  $\mathbf{B}, \mathbf{C} \in \mathcal{B}$  and  $d_{\mathbb{B}}(\mathbf{B}, \mathbf{C}) = 0$ , then  $f(\mathbf{B}) = f(\mathbf{C})$ . For each  $\mathbf{B}, \mathbf{C} \in \mathcal{B}$ , we have

$$\begin{aligned} f(\mathbf{B} \cup \mathbf{C}) &= f(\mathbf{B}) \cup f(\mathbf{C}), & f(\mathbf{B} \cap \mathbf{C}) &= f(\mathbf{B}) \cap f(\mathbf{C}), \\ \Omega \setminus f(\mathbf{B}) &= f(\Omega \setminus \mathbf{B}), & \mu(\mathbf{B}) &= \mu(f(\mathbf{B})). \end{aligned}$$

Moreover, the mapping  $f$  sends  $\mathcal{B}$  onto  $\widehat{\mathcal{B}}$ , because if  $\mathbf{C} \in \widehat{\mathcal{B}}$  then  $\mathbf{C} \in \mathcal{B}$  and  $f(\mathbf{C}) = \mathbf{C}$ . Therefore the mapping  $\widehat{f}$  that sends the equivalence class of each  $\mathbf{B} \in \mathcal{B}$  under  $d_{\mathbb{B}}$  to  $f(\mathbf{B})$  is well defined and is an isomorphism from the reduction of the pre-structure  $(\mathcal{B}, \sqcup, \sqcap, \neg, \top, \perp, \mu)$  onto the measured algebra  $(\widehat{\mathcal{B}}, \sqcup, \sqcap, \neg, \top, \perp, \mu)$ .

A model  $\mathcal{M}$  of  $T$  is  $\kappa^+$ -universal if every model of  $T$  of cardinality  $\leq \kappa$  is elementarily embeddable in  $\mathcal{M}$ . By Theorem 5.1.12 in [CK], every  $\kappa$ -saturated model of  $T$  is  $\kappa^+$ -universal, so  $\kappa^+$ -universal models of  $T$  exist. We now assume that  $\mathcal{M}$  is a  $\kappa^+$ -universal model of  $T$ , and prove that  $\mathcal{N}$  is isomorphic to the reduction of a nice randomization of  $\mathcal{M}$  with the underlying probability space  $(\Omega, \mathcal{B}, \mu)$ .

In the following paragraphs, we will use boldface letters  $\mathbf{b}, \mathbf{d}, \dots$  for elements of  $\widehat{\mathcal{K}}$ . Let  $L_{\widehat{\mathcal{K}}}$  be the first order signature formed by adding a constant symbol for each element  $\mathbf{b} \in \widehat{\mathcal{K}}$ . For each  $\omega \in \Omega$ , the set of  $L_{\widehat{\mathcal{K}}}$ -sentences

$$U(\omega) = \{\psi(\vec{\mathbf{b}}) : \omega \in \llbracket \psi(\vec{\mathbf{b}}) \rrbracket\}$$

is consistent with  $T$  and has cardinality  $\leq \kappa$ . By the Compactness and Löwenheim-Skolem theorems, each  $U(\omega)$  has a model  $(\mathcal{M}_\omega, \mathbf{b}_\omega)_{\mathbf{b} \in \widehat{\mathcal{K}}}$  of cardinality  $\leq \kappa$ . Since  $\mathcal{M}$  is  $\kappa^+$ -universal, for each  $\omega \in \Omega$  we may choose an elementary embedding  $h_\omega: \mathcal{M}_\omega \prec \mathcal{M}$ . Then  $(\mathcal{M}, h_\omega(\mathbf{b}_\omega))_{\mathbf{b} \in \widehat{\mathcal{K}}} \models U(\omega)$  for every  $\omega \in \Omega$ . It follows that for each formula  $\psi(\vec{v})$  of  $L$  and each tuple  $\vec{\mathbf{b}} \in \widehat{\mathcal{K}}^{<\mathbb{N}}$ ,

$$\llbracket \psi(\vec{\mathbf{b}}) \rrbracket = \{\omega \in \Omega : \mathcal{M}_\omega \models \psi(\vec{\mathbf{b}}_\omega)\} = \{\omega \in \Omega : \mathcal{M} \models \psi(h_\omega(\vec{\mathbf{b}}_\omega))\} \in \widehat{\mathcal{B}}.$$

For each formula  $\psi(\vec{v})$  of  $L$  and tuple  $\vec{c}$  of functions in  $M^\Omega$ , define

$$\llbracket \psi(\vec{c}) \rrbracket := \{\omega \in \Omega : \mathcal{M} \models \psi(\vec{c}(\omega))\}.$$

Let  $\mathcal{K}$  be the set of all functions  $a: \Omega \rightarrow M$  such that for some element  $\mathbf{b} \in \widehat{\mathcal{K}}$ , we have

$$\mu(\{\omega \in \Omega: a(\omega) = h_\omega(\mathbf{b}_\omega)\}) = 1.$$

We claim that for each  $a \in \mathcal{K}$  there is a unique element  $f(a) \in \widehat{\mathcal{K}}$  such that

$$\mu(\{\omega \in \Omega: a(\omega) = h_\omega(f(a)_\omega)\}) = 1.$$

The existence of  $f(a)$  is guaranteed by the definition of  $\mathcal{K}$ . To prove uniqueness, suppose  $\mathbf{b}, \mathbf{d} \in \widehat{\mathcal{K}}$  and

$$\mu(\{\omega \in \Omega: a(\omega) = h_\omega(\mathbf{b}_\omega)\}) = \mu(\{\omega \in \Omega: a(\omega) = h_\omega(\mathbf{d}_\omega)\}) = 1.$$

Then

$$\mu(\{\omega \in \Omega: h_\omega(\mathbf{b}_\omega) = h_\omega(\mathbf{d}_\omega)\}) = 1,$$

so

$$\mu(\llbracket \mathbf{b} = \mathbf{d} \rrbracket) = \mu(\{\omega \in \Omega: \mathbf{b}_\omega = \mathbf{d}_\omega\}) = 1,$$

and hence  $d_{\mathbb{K}}(\mathbf{b}, \mathbf{d}) = 0$ . Since  $d_{\mathbb{K}}$  is a metric on  $\widehat{\mathcal{K}}$ , it follows that  $\mathbf{b} = \mathbf{d}$ .

We now make some observations about the mapping  $f: \mathcal{K} \rightarrow \widehat{\mathcal{K}}$ . This mapping sends  $\mathcal{K}$  onto  $\widehat{\mathcal{K}}$ , because for each  $\mathbf{b} \in \widehat{\mathcal{K}}$ , we have  $f(a) = \mathbf{b}$  where  $a$  is the element of  $\mathcal{K}$  such that  $a(\omega) = h_\omega(\mathbf{b}_\omega)$  for all  $\omega \in \Omega$ . Suppose  $\vec{c} \in \mathcal{K}^{<\mathbb{N}}$  and  $\vec{\mathbf{d}} = f(\vec{c})$ . We have  $\vec{\mathbf{d}} \in \widehat{\mathcal{K}}^{<\mathbb{N}}$  and

$$\llbracket \psi(\vec{\mathbf{d}}) \rrbracket = \{\omega \in \Omega: \mathcal{M} \models \psi(h_\omega(\vec{\mathbf{d}}_\omega))\} \doteq \{\omega \in \Omega: \mathcal{M} \models \psi(\vec{c}(\omega))\} = \llbracket \psi(\vec{c}) \rrbracket.$$

Since the probability space  $(\Omega, \mathcal{B}, \mu)$  is complete,  $\llbracket \psi(\vec{\mathbf{d}}) \rrbracket \in \widehat{\mathcal{B}} \subseteq \mathcal{B}$ , and  $\llbracket \psi(\vec{\mathbf{d}}) \rrbracket \doteq \llbracket \psi(\vec{c}) \rrbracket$ , we have  $\llbracket \psi(\vec{c}) \rrbracket \in \mathcal{B}$  and  $\llbracket \psi(\vec{\mathbf{d}}) \rrbracket = f(\llbracket \psi(\vec{c}) \rrbracket)$ . Therefore, if  $a, c \in \mathcal{K}$  and  $d_{\mathbb{K}}(a, c) = 0$ , then  $d_{\mathbb{K}}(f(a), f(c)) = 0$ , and hence  $f(a) = f(c)$ . This shows that  $\mathcal{P} = (\mathcal{K}, \mathcal{B})$  is a well-defined pre-complete structure for  $L^R$ , and that the mapping  $\widehat{f}$  that sends the equivalence class of each  $\mathbf{B} \in \mathcal{B}$  to  $f(\mathbf{B})$ , and the equivalence class of each  $a \in \mathcal{K}$  to  $f(a)$ , is an isomorphism from the reduction of  $\mathcal{P}$  to  $\mathcal{N}$ .

It remains to show that  $\mathcal{P}$  is a nice randomization of  $\mathcal{M}$ . It is clear that  $\mathcal{P}$  satisfies conditions (1)-(3) in Definition 2.1.1.

Proof of (4): We have already shown that  $\llbracket \psi(\vec{c}) \rrbracket \in \mathcal{B}$  for each formula  $\psi(\vec{v})$  of  $L$  and each tuple  $\vec{c}$  in  $\mathcal{K}$ . For the other direction, let  $\mathbf{B} \in \mathcal{B}$ . By Corollary 2.1.6, there exist  $a, e \in \mathcal{K}$  such that  $\llbracket a \neq e \rrbracket \doteq \Omega$ . We may choose a function  $b \in M^\Omega$  such that  $b(\omega) = e(\omega)$  whenever  $a(\omega) \neq e(\omega)$ , and  $b(\omega) \neq a(\omega)$  for all  $\omega \in \Omega$ . Then  $b \in \mathcal{K}$  and  $\llbracket a \neq b \rrbracket = \Omega$ . By Lemma 2.1.8, there exists  $c \in \mathcal{K}$  which is a characteristic function of  $\mathbf{B}$  with respect to  $a, b$ . Then  $\llbracket c = a \rrbracket \doteq \mathbf{B}$ . Let  $d \in M^\Omega$  be the function such that  $d(\omega) = a(\omega)$  for  $\omega \in \mathbf{B}$ , and  $d(\omega) = b(\omega)$  for  $\omega \in \neg\mathbf{B}$ . Then  $\mu(\llbracket c = d \rrbracket) = 1$ , so  $d \in \mathcal{K}$ , and  $\llbracket a = d \rrbracket = \mathbf{B}$ . Thus (4) holds with  $\psi$  being the sentence  $a = d$ .

Proof of (5): Consider a formula  $\theta(u, \vec{v})$  of  $L$  and a tuple  $\vec{b}$  in  $\mathcal{K}$ . By Fullness, there exists  $c \in \mathcal{K}$  such that

$$\llbracket \theta(c, \vec{b}) \rrbracket \doteq \llbracket (\exists u)\theta(u, \vec{b}) \rrbracket.$$

We may choose a function  $a \in M^\Omega$  such that for all  $\omega \in \Omega$ ,

$$\mathcal{M} \models [\theta(c(\omega), \vec{b}(\omega)) \leftrightarrow (\exists u)\theta(u, \vec{b})] \text{ implies } a(\omega) = c(\omega),$$

and

$$\mathcal{M} \models [(\exists u)\theta(u, \vec{b}(\omega)) \rightarrow \theta(a(\omega), \vec{b}(\omega))].$$

Then  $\mu(\llbracket a = c \rrbracket) = 1$ , so  $a \in \mathcal{K}$  and

$$\llbracket \theta(a, \vec{b}) \rrbracket = \llbracket (\exists u)\theta(u, \vec{b}) \rrbracket,$$

as required.

Proof of (6) and (7): By Fact 2.1.4, the properties

$$(\forall x)(\forall y)d_{\mathbb{K}}(x, y) = \mu(\llbracket x \neq y \rrbracket), \quad (\forall \mathbf{U})(\forall \mathbf{V})d_{\mathbb{B}}(\mathbf{U}, \mathbf{V}) = \mu(\mathbf{U} \Delta \mathbf{V})$$

hold in some model of  $T^R$ . By Fact 2.1.3, these properties hold in all models of  $T^R$ , and thus in  $\mathcal{N}$ . Therefore (6) and (7) hold for  $\mathcal{P}$ .  $\square$

**2.2. Types and Definability.** For a first order structure  $\mathcal{M}$  and a set  $A$  of elements of  $\mathcal{M}$ ,  $\mathcal{M}_A$  denotes the structure formed by adding a new constant symbol to  $\mathcal{M}$  for each  $a \in A$ . The *type realized by a tuple  $\vec{b}$  over the parameter set  $A$  in  $\mathcal{M}$*  is the set  $\text{tp}^{\mathcal{M}}(\vec{b}/A)$  of formulas  $\varphi(\vec{u}, \vec{a})$  with  $\vec{a} \in A^{<\mathbb{N}}$  satisfied by  $\vec{b}$  in  $\mathcal{M}_A$ . We call  $\text{tp}^{\mathcal{M}}(\vec{b}/A)$  an *n-type* if  $n = |\vec{b}|$ .

In the following, let  $\mathcal{N}$  be a continuous structure and let  $A$  be a set of elements of  $\mathcal{N}$ .  $\mathcal{N}_A$  denotes the structure formed by adding a new constant symbol to  $\mathcal{N}$  for each  $a \in A$ . The *type  $\text{tp}^{\mathcal{N}}(\vec{b}/A)$  realized by  $\vec{b}$  over the parameter set  $A$  in  $\mathcal{N}$*  is the function  $p$  from formulas to  $[0, 1]$  such that for each formula  $\Phi(\vec{x}, \vec{a})$  with  $\vec{a} \in A^{<\mathbb{N}}$ , we have  $\Phi(\vec{x}, \vec{a})^p = \Phi(\vec{b}, \vec{a})^{\mathcal{N}}$ .

We now recall the notions of definable element and algebraic element from [BBHU]. An element  $b$  is *definable over  $A$*  in  $\mathcal{N}$ , in symbols  $b \in \text{dcl}^{\mathcal{N}}(A)$ , if there is a sequence of formulas  $\langle \Phi_k(x, \vec{a}_k) \rangle$  with  $\vec{a}_k \in A^{<\mathbb{N}}$  such that the sequence of functions  $\langle \Phi_k(x, \vec{a}_k)^{\mathcal{N}} \rangle$  converges uniformly in  $x$  to the distance function  $d(x, b)^{\mathcal{N}}$  of the corresponding sort.  $b$  is *algebraic over  $A$*  in  $\mathcal{N}$ , in symbols  $b \in \text{acl}^{\mathcal{N}}(A)$ , if there is a compact set  $C$  and a sequence of formulas  $\langle \Phi_k(x, \vec{a}_k) \rangle$  with  $\vec{a}_k \in A^{<\mathbb{N}}$  such that  $b \in C$  and the sequence of functions  $\langle \Phi_k(x, \vec{a}_k)^{\mathcal{N}} \rangle$  converges uniformly in  $x$  to the distance function  $d(x, C)^{\mathcal{N}}$  of the corresponding sort.

If the structure  $\mathcal{N}$  is clear from the context, we will sometimes drop the superscript and write  $\text{tp}, \text{dcl}, \text{acl}$  instead of  $\text{tp}^{\mathcal{N}}, \text{dcl}^{\mathcal{N}}, \text{acl}^{\mathcal{N}}$ .

**Fact 2.2.1.** ([BBHU], Exercises 10.7 and 10.10) For each element  $b$  of  $\mathcal{N}$ , the following are equivalent, where  $p = \text{tp}^{\mathcal{N}}(b/A)$ :

- (1)  $b$  is definable over  $A$  in  $\mathcal{N}$ ;
- (2) in each model  $\mathcal{N}' \succ \mathcal{N}$ ,  $b$  is the a unique element that realizes  $p$  over  $A$ ;
- (3)  $b$  is definable over some countable subset of  $A$  in  $\mathcal{N}$ .

**Fact 2.2.2.** ([BBHU], Exercise 10.8 and 10.11) For each element  $b$  of  $\mathcal{N}$ , the following are equivalent, where  $p = \text{tp}^{\mathcal{N}}(b/A)$ :

- (1)  $b$  is algebraic over  $A$  in  $\mathcal{N}$ ;
- (2) in each model  $\mathcal{N}' \succ \mathcal{N}$ , the set of elements  $b$  that realize  $p$  over  $A$  in  $\mathcal{N}'$  is compact.
- (3)  $b$  is algebraic over some countable subset of  $A$  in  $\mathcal{N}$ .

**Fact 2.2.3.** (Definable Closure, Exercises 10.10 and 10.11 in [BBHU])

- (1) If  $A \subseteq \mathcal{N}$  then  $\text{dcl}(A) = \text{dcl}(\text{dcl}(A))$  and  $\text{acl}(A) = \text{acl}(\text{acl}(A))$ .
- (2) If  $A$  is a dense subset of  $B$  and  $B \subseteq \mathcal{N}$ , then  $\text{dcl}(A) = \text{dcl}(B)$  and  $\text{acl}(A) = \text{acl}(B)$ .

It follows that for any  $A \subseteq \mathcal{N}$ ,  $\text{dcl}(A)$  and  $\text{acl}(A)$  are closed with respect to the metric in  $\mathcal{N}$ .

We now turn to the case where  $\mathcal{N}$  is a model of  $T^R$ . In that case, a set of elements of  $\mathcal{N}$  may contain elements of both sorts  $\mathbb{K}, \mathbb{B}$ . But as we will now explain, we need only consider definability over sets of parameters of sort  $\mathbb{K}$ .

**Remark 2.2.4.** Let  $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$  be a model of  $T^R$ . Since every model of  $T$  has at least two elements,  $\mathcal{N}$  has a pair of elements  $a, b$  of sort  $\mathbb{K}$  such that  $\mathcal{N} \models \llbracket a = b \rrbracket = \perp$ . For each event  $\mathbb{D} \in \widehat{\mathcal{B}}$ , let  $1_{\mathbb{D}}$  be the characteristic function of  $\mathbb{D}$  with respect to  $a, b$ . Then in the model  $\mathcal{N}$ ,  $\mathbb{D}$  is definable over  $\{a, b, 1_{\mathbb{D}}\}$ , and  $1_{\mathbb{D}}$  is definable over  $\{a, b, \mathbb{D}\}$ .

*Proof.* By Fact 2.2.1. □

In view of Remark 2.2.4 and Fact 2.2.3, if  $C$  is a set of parameters in  $\mathcal{N}$  of both sorts, and there are elements  $a, b \in C$  such that  $\mathcal{N} \models \llbracket a = b \rrbracket = \perp$ , then an element of either sort is definable over  $C$  if and only if it is definable over the set of parameters of sort  $\mathbb{K}$  obtained by replacing each element of  $C$  of sort  $\mathbb{B}$  by its characteristic function with respect to  $a, b$ . For this reason, in a model  $\mathcal{N}$  of  $T^R$  we will only consider definability over sets of parameters of sort  $\mathbb{K}$ . We write  $\text{dcl}_{\mathbb{B}}(A)$  for the set of elements of sort  $\mathbb{B}$  that are definable over  $A$  in  $\mathcal{N}$ , and write  $\text{dcl}(A)$  for the set of elements of sort  $\mathbb{K}$  that are definable over  $A$  in  $\mathcal{N}$ . Similarly for  $\text{acl}_{\mathbb{B}}(A)$  and  $\text{acl}(A)$ .

**2.3. Conventions and Notation.** We will assume hereafter that  $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$  is a model of  $T^R$ ,  $\mathcal{P} = (\mathcal{K}, \mathcal{B})$  is a nice randomization of a model  $\mathcal{M} \models T$  with probability space  $(\Omega, \mathcal{B}, \mu)$ , and  $\mathcal{N}$  is the reduction of  $\mathcal{P}$ . The existence of  $\mathcal{P}$  is guaranteed by Proposition 2.1.10.

We will use boldfaced letters  $\mathbf{a}, \mathbf{b}, \dots$  for elements of  $\widehat{\mathcal{K}}$ . For each element  $\mathbf{a} \in \widehat{\mathcal{K}}$ , we will choose once and for all an element  $a \in \mathcal{K}$  such that the image of  $a$  under the reduction map is  $\mathbf{a}$ . It follows that for each first order formula  $\varphi(\vec{v})$ ,  $\llbracket \varphi(\vec{\mathbf{a}}) \rrbracket$  is the image of  $\llbracket \varphi(\vec{a}) \rrbracket$  under the reduction map. For any countable set  $A \subseteq \widehat{\mathcal{K}}$  and each  $\omega \in \Omega$ , we define

$$A(\omega) = \{a(\omega) : \mathbf{a} \in A\}.$$

When  $A \subseteq \widehat{\mathcal{K}}$ ,  $\text{cl}(A)$  denotes the closure of  $A$  in the metric  $d_{\mathbb{K}}$ . When  $B \subseteq \widehat{\mathcal{B}}$ ,  $\text{cl}(B)$  denotes the closure of  $B$  in the metric  $d_{\mathbb{B}}$ , and  $\sigma(B)$  denotes the smallest  $\sigma$ -subalgebra of  $\widehat{\mathcal{B}}$  containing  $B$ .

## 3. RANDOMIZATIONS OF ARBITRARY THEORIES

3.1. **Definability in Sort  $\mathbb{B}$ .** We characterize the set of elements of  $\widehat{\mathcal{B}}$  that are definable in  $\mathcal{N}$  over a set of parameters  $A \subseteq \widehat{\mathcal{K}}$ .

**Definition 3.1.1.** For each  $A \subseteq \widehat{\mathcal{K}}$ , we say that an event  $E$  is *first order definable* over  $A$ , in symbols  $E \in \text{fdcl}_{\mathbb{B}}(A)$ , if  $E = \llbracket \varphi(\vec{a}) \rrbracket$  for some first order formula  $\varphi(\vec{v})$  and tuple  $\vec{a}$  in  $A^{<\mathbb{N}}$ .

**Theorem 3.1.2.** For each  $A \subseteq \widehat{\mathcal{K}}$ ,  $\text{dcl}_{\mathbb{B}}(A) = \text{cl}(\text{fdcl}_{\mathbb{B}}(A)) = \sigma(\text{fdcl}_{\mathbb{B}}(A))$ .

*Proof.* By quantifier elimination (Fact 2.1.7), in any elementary extension  $\mathcal{N}' \succ \mathcal{N}$ , two events have the same type over  $A$  if and only if they have the same type over  $\text{fdcl}_{\mathbb{B}}(A)$ . Then by Fact 2.2.1,  $\text{dcl}_{\mathbb{B}}(A) = \text{dcl}_{\mathbb{B}}(\text{fdcl}_{\mathbb{B}}(A))$ . Moreover,  $\text{dcl}_{\mathbb{B}}(\text{fdcl}_{\mathbb{B}}(A))$  is equal to the definable closure of  $\text{fdcl}_{\mathbb{B}}(A)$  in the pure measured algebra  $(\widehat{\mathcal{B}}, \mu)$ . By Observation 16.7 in [BBHU], the definable closure of  $\text{fdcl}_{\mathbb{B}}(A)$  in  $(\widehat{\mathcal{B}}, \mu)$  is equal to  $\sigma(\text{fdcl}_{\mathbb{B}}(A))$ , so  $\text{dcl}_{\mathbb{B}}(A) = \sigma(\text{fdcl}_{\mathbb{B}}(A))$ . Since  $\text{fdcl}_{\mathbb{B}}(A)$  is a Boolean subalgebra of  $\widehat{\mathcal{B}}$ ,  $\text{cl}(\text{fdcl}_{\mathbb{B}}(A))$  is a Boolean subalgebra of  $\widehat{\mathcal{B}}$ . By metric completeness,  $\text{cl}(\text{fdcl}_{\mathbb{B}}(A))$  is a  $\sigma$ -algebra and  $\sigma(\text{fdcl}_{\mathbb{B}}(A))$  is closed, so  $\text{cl}(\text{fdcl}_{\mathbb{B}}(A)) = \sigma(\text{fdcl}_{\mathbb{B}}(A))$ .  $\square$

**Corollary 3.1.3.** The only events that are definable without parameters in  $\mathcal{N}$  are  $\top$  and  $\perp$ .

*Proof.* For every first order sentence  $\varphi$ , either  $T \models \varphi$  and  $T^R \models \llbracket \varphi \rrbracket = \top$ , or  $T \models \neg\varphi$  and  $T^R \models \llbracket \varphi \rrbracket = \perp$ . So  $\text{fdcl}_{\mathbb{B}}(\emptyset) = \{\top, \perp\}$ .  $\square$

3.2. **First Order and Pointwise Definability.** To prepare the way for a characterization of the definable elements of sort  $\mathbb{K}$ , we introduce two auxiliary notions, one that is stronger than definability in sort  $\mathbb{K}$  and one that is weaker than definability in sort  $\mathbb{K}$ . We will work in the nice randomization  $\mathcal{P} = (\mathcal{K}, \mathcal{B})$  of  $\mathcal{M}$ , and let  $A$  be a subset of  $\widehat{\mathcal{K}}$  and  $\mathbf{b}$  be an element of  $\widehat{\mathcal{K}}$ .

**Definition 3.2.1.** A first order formula  $\varphi(u, \vec{v})$  is *functional* if

$$T \models (\forall \vec{v})(\exists^{\leq 1} u)\varphi(u, \vec{v}).$$

We say that  $\mathbf{b}$  is *first order definable on  $E$  over  $A$*  if there is a functional formula  $\varphi(u, \vec{v})$  and a tuple  $\vec{a} \in A^{<\mathbb{N}}$  such that  $E = \llbracket \varphi(\mathbf{b}, \vec{a}) \rrbracket$ .

We say that  $\mathbf{b}$  is *first order definable over  $A$* , in symbols  $\mathbf{b} \in \text{fdcl}(A)$ , if  $\mathbf{b}$  is first order definable on  $\top$  over  $A$ .

**Remarks 3.2.2.**  $\mathbf{b}$  is first order definable over  $A$  if and only if there is a first order formula  $\varphi(u, \vec{v})$  and a tuple  $\vec{a}$  from  $A$  such that

$$\mu(\llbracket (\forall u)(\varphi(u, \vec{a}) \leftrightarrow u = \mathbf{b}) \rrbracket) = 1.$$

First order definability has finite character, that is,  $\mathbf{b}$  is first order definable over  $A$  if and only if  $\mathbf{b}$  is first order definable over some finite subset of  $A$ .

If  $\mathbf{b}$  is first order definable on  $E$  over  $A$ , then  $E$  is first order definable over  $A \cup \{\mathbf{b}\}$ .

If  $\mathbf{b}$  is first order definable on  $\mathbf{D}$  over  $A$ , and  $\mathbf{E}$  is first order definable over  $A \cup \{\mathbf{b}\}$ , then  $\mathbf{b}$  is first order definable on  $\mathbf{D} \sqcap \mathbf{E}$  over  $A$ .

**Lemma 3.2.3.** *If  $\mathbf{b}$  is first order definable over  $A$  then  $\mathbf{b}$  is definable over  $A$  in  $\mathcal{N}$ . Thus  $\text{fdcl}(A) \subseteq \text{dcl}(A)$ .*

*Proof.* Let  $\mathcal{N}' \succ \mathcal{N}$  and suppose that  $\text{tp}^{\mathcal{N}'}(\mathbf{b}) = \text{tp}^{\mathcal{N}'}(\mathbf{d})$ . Then

$$\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket = \llbracket \varphi(\mathbf{d}, \vec{\mathbf{a}}) \rrbracket = \top.$$

Since  $\varphi$  is functional,

$$\llbracket (\forall t)(\forall u)(\varphi(t, \vec{\mathbf{a}}) \wedge \varphi(u, \vec{\mathbf{a}}) \rightarrow t = u) \rrbracket = \top.$$

Then  $\llbracket \mathbf{b} = \mathbf{d} \rrbracket = \top$ , so  $\mathbf{b} = \mathbf{d}$ , and by Fact 2.2.1,  $\mathbf{b} \in \text{dcl}(A)$ .  $\square$

**Definition 3.2.4.** When  $A$  is countable, we define

$$\llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket := \{\omega \in \Omega : b(\omega) \in \text{dcl}^{\mathcal{M}}(A(\omega))\}.$$

**Lemma 3.2.5.** *If  $A$  is countable, then*

$$\llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket = \bigcup \{ \llbracket \theta(b, \vec{\mathbf{a}}) \rrbracket : \theta(u, \vec{v}) \text{ functional, } \vec{\mathbf{a}} \in A^{<\mathbb{N}} \},$$

and  $\llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket \in \mathcal{B}$ .

*Proof.* Note that for every first order formula  $\theta(u, \vec{v})$ , the formula

$$\theta(u, \vec{v}) \wedge (\exists^{\leq 1} u) \theta(u, \vec{v})$$

is functional. Therefore  $\omega \in \llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket$  if and only if  $b(\omega) \in \text{dcl}^{\mathcal{M}}(A(\omega))$ , and this holds if and only if there is a functional formula  $\theta(u, \vec{v})$  and a tuple  $\vec{\mathbf{a}} \in A^{<\mathbb{N}}$  such that  $\mathcal{M} \models \theta(b(\omega), \vec{\mathbf{a}}(\omega))$ . Since  $A$  and  $L$  are countable,  $\llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket$  is the union of countably many events in  $\mathcal{B}$ , and thus belongs to  $\mathcal{B}$ .  $\square$

**Definition 3.2.6.** When  $A$  is countable, we say that  $\mathbf{b}$  is *pointwise definable over  $A$*  if

$$\mu(\llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket) = 1.$$

**Corollary 3.2.7.** *If  $A$  is countable, then  $\mathbf{b}$  is pointwise definable over  $A$  if and only if there is a function  $f$  on  $\Omega$  such that:*

- (1) *For each  $\omega \in \Omega$ ,  $f(\omega)$  is a pair  $\langle \theta_\omega(u, \vec{v}), \vec{\mathbf{a}}_\omega \rangle$  where  $\theta_\omega(u, \vec{v})$  is functional and  $\vec{\mathbf{a}}_\omega \in A^{|\vec{v}|}$ ;*
- (2)  *$f$  is  $\sigma(\text{fdcl}_{\mathbb{B}}(A))$ -measurable (i.e., the inverse image of each point belongs to  $\sigma(\text{fdcl}_{\mathbb{B}}(A))$ );*
- (3)  *$\mathcal{M} \models \theta_\omega(b(\omega), \vec{\mathbf{a}}_\omega(\omega))$  for almost every  $\omega \in \Omega$ .*

*Proof.* If  $\omega \in \llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket$ , let  $f(\omega)$  be the first pair  $\langle \theta_\omega, \vec{\mathbf{a}}_\omega \rangle$  such that  $\theta_\omega(u, \vec{v})$  is functional,  $\vec{\mathbf{a}}_\omega \in A^{|\vec{v}|}$ , and  $\mathcal{M} \models \theta_\omega(b(\omega), \vec{\mathbf{a}}_\omega(\omega))$ . Otherwise let  $f(\omega) = \langle \perp, \emptyset \rangle$ . The result then follows from Lemma 3.2.5.  $\square$

**Lemma 3.2.8.** *If  $\mathbf{b}$  is definable over  $A$  in  $\mathcal{N}$ , then  $\mathbf{b}$  is pointwise definable over some countable subset of  $A$ .*

*Proof.* By Fact 2.2.1 (3), we may assume that  $A$  is countable. By Lemma 3.2.5, the measure  $r := \mu(\llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket)$  exists. Suppose  $\mathbf{b}$  is not pointwise definable over  $A$ . Then  $r < 1$ . For each finite collection  $\chi_1(u, \vec{v}), \dots, \chi_n(u, \vec{v})$  of first order formulas, each tuple  $\vec{\mathbf{a}} \in A^{<\mathbb{N}}$ , and each  $\omega \in \Omega \setminus \llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket$ , the sentence

$$(\exists u)[u \neq b(\omega) \wedge \bigwedge_{i=1}^n [\chi_i(b(\omega), \vec{\mathbf{a}}(\omega)) \leftrightarrow \chi_i(u, \vec{\mathbf{a}}(\omega))]$$

holds in  $\mathcal{M}$ , because  $b(\omega)$  is not definable over  $A(\omega)$ . Therefore in  $\mathcal{P}$  we have

$$\mu(\llbracket (\exists u)[u \neq b \wedge \bigwedge_{i=1}^n [\chi_i(b, \vec{\mathbf{a}}) \leftrightarrow \chi_i(u, \vec{\mathbf{a}})] \rrbracket) \geq 1 - r.$$

By condition 2.1.1 (5), there is an element  $\mathbf{d} \in \widehat{\mathcal{K}}$  such that

$$\mu(\llbracket \mathbf{d} \neq b \wedge \bigwedge_{i=1}^n [\chi_i(b, \vec{\mathbf{a}}) \leftrightarrow \chi_i(\mathbf{d}, \vec{\mathbf{a}})] \rrbracket) \geq 1 - r.$$

It follows that  $\mu(\llbracket \mathbf{d} \neq \mathbf{b} \rrbracket) \geq 1 - r$ , and  $\llbracket \chi_i(b, \vec{\mathbf{a}}) \rrbracket \doteq \llbracket \chi_i(\mathbf{d}, \vec{\mathbf{a}}) \rrbracket$  for each  $i \leq n$ . By compactness, in some elementary extension of  $\mathcal{N}$  there is an element  $\mathbf{d}$  such that  $\mu(\llbracket \mathbf{d} \neq \mathbf{b} \rrbracket) \geq 1 - r$ , and  $\llbracket \chi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket = \llbracket \chi(\mathbf{d}, \vec{\mathbf{a}}) \rrbracket$  for each first order formula  $\chi(u, \vec{v})$ . Then  $\mathbf{d} \neq \mathbf{b}$ , and by quantifier elimination,  $\text{tp}(\mathbf{d}/A) = \text{tp}(\mathbf{b}/A)$ . Hence by Fact 2.2.1 (2),  $\mathbf{b} \notin \text{dcl}(A)$ .  $\square$

The following example shows that the converse of Lemma 3.2.8 fails badly.

**Example 3.2.9.** Let  $\mathcal{M}$  be a finite structure with a constant symbol for every element. Then every element of  $\mathcal{K}$  is pointwise definable without parameters, but the only elements of  $\widehat{\mathcal{K}}$  that are definable without parameters are the equivalence classes of constant functions  $b: \Omega \rightarrow \mathcal{M}$ .

**3.3. Definability in Sort  $\mathbb{K}$ .** We will now give necessary and sufficient conditions for an element of  $\mathbf{b} \in \widehat{\mathcal{K}}$  to be definable over a parameter set  $A \subseteq \widehat{K}$  in  $\mathcal{N}$ .

**Theorem 3.3.1.**  *$\mathbf{b}$  is definable over  $A$  if and only if there exist pairwise disjoint events  $\{\mathbf{E}_n: n \in \mathbb{N}\}$  such that  $\sum_{n \in \mathbb{N}} \mu(\mathbf{E}_n) = 1$ , and for each  $n$ ,  $\mathbf{E}_n$  is definable over  $A$ , and  $\mathbf{b}$  is first order definable on  $\mathbf{E}_n$  over  $A$ .*

*Proof.* ( $\Rightarrow$ ): Suppose  $\mathbf{b} \in \text{dcl}(A)$ . By Lemma 3.2.8,  $\mathbf{b}$  is pointwise definable over some countable subset  $A_0$  of  $A$ . The set of all events  $\mathbf{C}$  such that  $\mathbf{b}$  is first order definable on  $\mathbf{C}$  over  $A_0$  is countable, and may be arranged in a list  $\{\mathbf{C}_n: n \in \mathbb{N}\}$ . Let  $\mathbf{E}_0 = \mathbf{C}_0$ , and

$$\mathbf{E}_{n+1} = \mathbf{C}_{n+1} \sqcap \neg(\mathbf{C}_0 \sqcup \dots \sqcup \mathbf{C}_n).$$

The events  $\mathbf{E}_n$  are pairwise disjoint, and for each  $n$  we have

$$\mathbf{E}_0 \sqcup \dots \sqcup \mathbf{E}_n = \mathbf{C}_0 \sqcup \dots \sqcup \mathbf{C}_n.$$

By Remarks 3.2.2, for each  $n$ ,  $\mathbf{b}$  is first order definable on  $\mathbf{E}_n$  over  $A$ . By Lemma 3.2.5 and pointwise definability,

$$\sum_{n \in \mathbb{N}} \mu(\mathbf{E}_n) = \lim_{n \rightarrow \infty} \mu(\mathbf{C}_0 \sqcup \cdots \sqcup \mathbf{C}_n) = \mu(\llbracket \text{dcl}^{\mathcal{M}}(A_0) \rrbracket) = 1.$$

By Remarks 3.2.2,  $\mathbf{E}_n$  is definable over  $A \cup \{\mathbf{b}\}$ , and since  $\mathbf{b}$  is definable over  $A$ ,  $\mathbf{E}_n$  is definable over  $A$  by Fact 2.2.3.

( $\Leftarrow$ ): Let  $\mathbf{E}_n$  be as in the theorem. For each  $n$ , we have  $\mathbf{E}_n = \llbracket \theta_n(\mathbf{b}, \vec{\mathbf{a}}_n) \rrbracket$  for some functional formula  $\theta_n$  and tuple  $\vec{\mathbf{a}}_n \in A^{<\mathbb{N}}$ . Since  $\mathbf{E}_n$  is definable over  $A$ , by Theorem 3.1.2 there is a sequence of formulas  $\psi_k(\vec{v})$  and tuples  $\vec{\mathbf{a}}_k \in A^{<\mathbb{N}}$  such that

$$\lim_{k \rightarrow \infty} d_{\mathbb{B}}(\llbracket \psi_k(\vec{\mathbf{a}}_k) \rrbracket, \llbracket \theta_n(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket) = 0.$$

Suppose  $\mathbf{d}$  has the same type over  $A$  as  $\mathbf{b}$  in some elementary extension  $\mathcal{N}'$  of  $\mathcal{N}$ . Then

$$\lim_{k \rightarrow \infty} d_{\mathbb{B}}(\llbracket \psi_k(\vec{\mathbf{a}}_k) \rrbracket, \llbracket \theta_n(\mathbf{d}, \vec{\mathbf{a}}) \rrbracket) = 0.$$

Hence

$$\llbracket \theta_n(\mathbf{d}, \vec{\mathbf{a}}_n) \rrbracket = \llbracket \theta_n(\mathbf{b}, \vec{\mathbf{a}}_n) \rrbracket = \mathbf{E}_n$$

in  $\mathcal{N}'$ . Since  $\theta_n(u, \vec{v})$  is functional, we have  $\llbracket \theta_n(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \sqsubseteq \llbracket \mathbf{d} = \mathbf{b} \rrbracket$  for each  $n$ . Then

$$\mu(\llbracket \mathbf{d} = \mathbf{b} \rrbracket) \geq \sum_{n \in \mathbb{N}} \mu(\mathbf{E}_n) = 1,$$

so  $\mathbf{d} = \mathbf{b}$ . Then by Fact 2.2.1,  $\mathbf{b} \in \text{dcl}(A)$ .  $\square$

**Corollary 3.3.2.** *An element  $\mathbf{b} \in \widehat{\mathcal{K}}$  is definable without parameters if and only if  $\mathbf{b}$  is first order definable without parameters. Thus  $\text{dcl}(\emptyset) = \text{fdcl}(\emptyset)$ .*

*Proof.* ( $\Rightarrow$ ): Suppose  $\mathbf{b} \in \text{dcl}(\emptyset)$ . By Theorem 3.3.1, there is an event  $\mathbf{E}$  such that  $\mu(\mathbf{E}) > 0$ ,  $\mathbf{E}$  is definable without parameters, and  $\mathbf{b}$  is first order definable on  $\mathbf{E}$  without parameters. By Corollary 3.1.3 we have  $\mathbf{E} = \top$ , so  $\mathbf{b}$  is first order definable without parameters.

( $\Leftarrow$ ): By Lemma 3.2.3.  $\square$

**Corollary 3.3.3.** *If  $\text{fdcl}_{\mathbb{B}}(A)$  is finite, then  $\text{dcl}_{\mathbb{B}}(A) = \text{fdcl}_{\mathbb{B}}(A)$  and  $\text{dcl}(A) = \text{fdcl}(A)$ .*

*Proof.*  $\text{dcl}_{\mathbb{B}}(A) = \text{fdcl}_{\mathbb{B}}(A)$  follows from Theorem 3.1.2. Lemma 3.2.3 gives  $\text{dcl}(A) \supseteq \text{fdcl}(A)$ . For the other inclusion, suppose  $\mathbf{b} \in \text{dcl}(A)$ . By Theorem 3.3.1, there is a finite partition  $\mathbf{E}_0, \dots, \mathbf{E}_k$  of  $\top$ , a tuple  $\vec{\mathbf{a}} \in A^{<\mathbb{N}}$ , and first order formulas  $\psi_i(\vec{v})$  such that  $\mathbf{E}_i = \llbracket \psi_i(\vec{\mathbf{a}}) \rrbracket$  and  $\mathbf{b}$  is first order definable on  $\mathbf{E}_i$ . Then there are functional formulas  $\varphi_i(u, \vec{v})$  such that  $\mathbf{E}_i \doteq \llbracket \varphi_i(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket$ . We may take the formulas  $\psi_i(\vec{v})$  to be pairwise inconsistent and such that

$T \models \bigvee_{i=0}^n \psi(\vec{v})$ . Then  $\bigwedge_{i=0}^n (\psi_i(\vec{v}) \rightarrow \varphi_i(u, \vec{v}))$  is a functional formula such that

$$\llbracket \bigwedge_{i=0}^n (\psi_i(\vec{a}) \rightarrow \varphi_i(\mathbf{b}, \vec{a})) \rrbracket = \top,$$

so  $\mathbf{b}$  is first order definable over  $A$ .  $\square$

**Corollary 3.3.4.**  *$\mathbf{b}$  is definable over  $A$  if and only if:*

- (1)  $\mathbf{b}$  is pointwise definable over some countable subset of  $A$ ;
- (2) for each functional formula  $\varphi(u, \vec{v})$  and tuple  $\vec{a} \in A^{<\mathbb{N}}$ ,  $\llbracket \varphi(\mathbf{b}, \vec{a}) \rrbracket$  is definable over  $A$ .

*Proof.* ( $\Rightarrow$ ): Suppose  $\mathbf{b} \in \text{dcl}(A)$ . Then (1) holds by Lemma 3.2.8.  $\llbracket \varphi(\mathbf{b}, \vec{a}) \rrbracket$  is obviously definable over  $A \cup \{\mathbf{b}\}$ , so  $\llbracket \varphi(\mathbf{b}, \vec{a}) \rrbracket$  is definable over  $A$  by Fact 2.2.3, and thus (2) holds.

( $\Leftarrow$ ): Assume conditions (1) and (2). By (1) and Lemma 3.2.5, there is a sequence of functional formulas  $\theta_n(u, \vec{v})$  and tuples  $\vec{a}_n \in A^{<\mathbb{N}}$  such that

$$\llbracket \mathbf{b} \in \text{dcl}^{\mathcal{M}}(A) \rrbracket = \bigcup_{n \in \mathbb{N}} \llbracket \theta_n(\mathbf{b}, \vec{a}_n) \rrbracket \doteq \Omega.$$

Let  $\mathbf{E}_n = \llbracket \theta_n(\mathbf{b}, \vec{a}_n) \rrbracket$ , so  $\mathbf{b}$  is first order definable on  $\mathbf{E}_n$  over  $A$ . By Remark 3.2.2, we may take the  $\mathbf{E}_n$  to be pairwise disjoint, and thus  $\sum_{n \in \mathbb{N}} \mu(\mathbf{E}_n) = 1$ . By (2),  $\mathbf{E}_n$  is definable over  $A$  for each  $n$ . Then by Theorem 3.3.1,  $\mathbf{b} \in \text{dcl}(A)$ .  $\square$

**Corollary 3.3.5.**  *$\mathbf{b}$  is definable over  $A$  if and only if:*

- (1)  $\mathbf{b}$  is pointwise definable over some countable subset of  $A$ ;
- (2)  $\text{fdcl}_{\mathbb{B}}(A \cup \{\mathbf{b}\}) \subseteq \text{dcl}_{\mathbb{B}}(A)$ .

**Theorem 3.3.6.**  *$\mathbf{b}$  is definable over  $A$  if and only if  $\mathbf{b} = \lim_{m \rightarrow \infty} \mathbf{b}_m$ , where each  $\mathbf{b}_m$  is first-order definable over  $A$ . Thus  $\text{dcl}(A) = \text{cl}(\text{fdcl}(A))$ .*

*Proof.* ( $\Rightarrow$ ): Suppose that  $\mathbf{b} \in \text{dcl}(A)$ . If  $A$  is empty, then  $\mathbf{b}$  is already first order definable from  $A$  by Corollary 3.3.2. Assume  $A$  is not empty and let  $\mathbf{c} \in A$ . Let  $\{\mathbf{E}_n : n \in \mathbb{N}\}$  be as in Theorem 3.3.1, and fix an  $\varepsilon > 0$ . Then for some  $n$ ,  $\sum_{k=0}^n \mu(\mathbf{E}_k) > 1 - \varepsilon$ . For each  $k$ ,  $\mathbf{E}_k$  is definable over  $A$ , so by Theorem 3.1.2, there is an event  $\mathbf{D}_k \in \text{fdcl}_{\mathbb{B}}(A)$  such that  $\mu(\mathbf{D}_k \triangle \mathbf{E}_k) < \varepsilon/n$ . Since the events  $\mathbf{E}_k$  are pairwise disjoint, we may also take the events  $\mathbf{D}_k$  to be pairwise disjoint. We have  $\mathbf{E}_k = \llbracket \theta_k(\mathbf{b}, \vec{a}_k) \rrbracket$  for some functional  $\theta_k(u, \vec{v})$ , so we may assume that  $\mathbf{D}_k$  has the additional properties that  $\mathbf{D}_k \sqsubseteq \llbracket (\exists! u) \theta_k(u, \vec{a}_k) \rrbracket$ , and that  $\mathbf{D}_k = \llbracket \psi_k(\vec{a}_k) \rrbracket$  for some formula  $\psi_k(\vec{v})$ . Then there is a unique element  $\mathbf{d} \in \widehat{\mathcal{K}}$  such that

$$\begin{cases} \mathcal{M} \models \theta_k(\mathbf{d}(\omega), \vec{a}_k(\omega)) & \text{if } k \leq n \text{ and } \omega \in \llbracket \psi_k(\vec{a}_k) \rrbracket, \\ \mathbf{d}(\omega) = \mathbf{c}(\omega) & \text{if } \omega \in \Omega \setminus \bigcup_{k=0}^n \llbracket \psi_k(\vec{a}_k) \rrbracket. \end{cases}$$

Then  $\mathbf{d}$  is first order definable over  $A$ , and  $d_{\mathbb{K}}(\mathbf{b}, \mathbf{d}) < \varepsilon$ .

( $\Leftarrow$ ): This follows because first order definability implies definability (Lemma 3.2.3) and the set  $\text{dcl}(A)$  is metrically closed (Fact 2.2.3 (2)).  $\square$

The following result was proved in [Be] by an indirect argument using Lascar types. We give a simple direct proof here.

**Proposition 3.3.7.** *For any model  $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$  of  $T^R$  and set  $A \subseteq \widehat{\mathcal{K}}$ ,  $\text{acl}_{\mathbb{B}}(A) = \text{dcl}_{\mathbb{B}}(A)$  and  $\text{acl}(A) = \text{dcl}(A)$ .*

*Proof.* By Facts 2.2.1 and 2.2.2, we may assume  $\mathcal{N}$  is  $\aleph_1$ -saturated and  $A$  is countable. Suppose an event  $E \in \widehat{\mathcal{B}}$  is not definable over  $A$ . By Fact 2.2.1 and  $\aleph_1$ -saturation there exists  $D \in \widehat{\mathcal{B}}$  such that  $\text{tp}(D/A) = \text{tp}(E/A)$  but  $d_{\mathbb{B}}(D, E) > 0$ . By  $\aleph_1$ -saturation again, there is a countable sequence of events  $\langle F_n : n \in \mathbb{N} \rangle$  in  $\widehat{\mathcal{B}}$  such that

$$\mu(C \cap F_n) = \mu(C \setminus F_n) = \mu(C)/2$$

for each  $n$  and each event  $C$  in the Boolean algebra generated by

$$\text{fdcl}_{\mathbb{B}}(A) \cup \{D, E\} \cup \{F_k : k < n\}.$$

For each  $n$ , let

$$D_n = (D \cap F_n) \cup (E \setminus F_n).$$

Then for each  $C \in \text{fdcl}_{\mathbb{B}}(A)$  and  $n \in \mathbb{N}$ , we have

$$\mu(D_n \cap C) = \mu(D \cap C)/2 + \mu(E \cap C)/2 = \mu(E \cap C).$$

By quantifier elimination,  $\text{tp}(D_n/A) = \text{tp}(E/A)$  for each  $n \in \mathbb{N}$ . Moreover, whenever  $k < n$  we have

$$D_n \setminus D_k = ((D \setminus D_k) \cap F_n) \cup ((E \setminus D_k) \setminus F_n),$$

so

$$\mu(D_n \setminus D_k) = \mu(D \setminus D_k)/2 + \mu(E \setminus D_k)/2.$$

Note that whenever  $\text{tp}(D'/A) = \text{tp}(D''/A)$ , we have  $\mu(D') = \mu(D'')$ , and hence

$$\mu(D' \setminus D'') = \mu(D'' \setminus D') = d_{\mathbb{B}}(D', D'')/2.$$

Therefore

$$d_{\mathbb{B}}(D_n, D_k) = d_{\mathbb{B}}(D, D_k)/2 + d_{\mathbb{B}}(E, D_k)/2 \geq d_{\mathbb{B}}(D, E)/2.$$

It follows that the set of realizations of  $\text{tp}(E/A)$  is not compact, and  $E$  is not algebraic over  $A$ . This shows that  $\text{acl}_{\mathbb{B}}(A) = \text{dcl}_{\mathbb{B}}(A)$ .

Now suppose  $\mathbf{b} \in \text{acl}(A) \setminus \text{dcl}(A)$ . There is an element  $\mathbf{c} \in \widehat{\mathcal{K}}$  such that  $\text{tp}(\mathbf{b}/A) = \text{tp}(\mathbf{c}/A)$  but  $d_{\mathbb{K}}(\mathbf{b}, \mathbf{c}) > 0$ . For each first order formula  $\psi(u, \vec{v})$  and  $\vec{\mathbf{a}} \in A^{<\mathbb{N}}$ ,  $\llbracket \psi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \in \text{acl}_{\mathbb{B}}(\{\mathbf{b}\} \cup A) \subseteq \text{acl}_{\mathbb{B}}(\text{acl}(A))$ . By Fact 2.2.3,  $\llbracket \psi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \in \text{acl}_{\mathbb{B}}(A)$ . By the preceding paragraph,  $\llbracket \psi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \in \text{dcl}_{\mathbb{B}}(A)$ . Since  $\text{tp}(\mathbf{b}/A) = \text{tp}(\mathbf{c}/A)$ , we have  $\text{tp}(\llbracket \psi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket/A) = \text{tp}(\llbracket \psi(\mathbf{c}, \vec{\mathbf{a}}) \rrbracket/A)$ . By Fact 2.2.1, it follows that  $\llbracket \psi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket = \llbracket \psi(\mathbf{c}, \vec{\mathbf{a}}) \rrbracket$  for every first order formula  $\psi(u, \vec{v})$ . Then  $\text{tp}(b(\omega)/A(\omega)) = \text{tp}(c(\omega)/A(\omega))$  for  $\mu$ -almost all  $\omega$ . By  $\aleph_1$ -saturation, there are countably many independent events  $D_n \in \widehat{\mathcal{B}}$  such that  $D_n \subseteq \llbracket \mathbf{b} \neq \mathbf{c} \rrbracket$  and  $\mu(D_n) = d_{\mathbb{K}}(\mathbf{b}, \mathbf{c})/2$ . Let  $\mathbf{c}_n$  agree with  $\mathbf{c}$  on  $D_n$  and agree with  $\mathbf{b}$  elsewhere. We have  $\text{tp}(\mathbf{c}_n/A) = \text{tp}(\mathbf{b}/A)$  for every  $n \in \mathbb{N}$ , and

$d_{\mathbb{K}}(\mathbf{c}_n, \mathbf{c}_k) = d_{\mathbb{K}}(\mathbf{b}, \mathbf{c})/2$  whenever  $k < n$ . Thus the set of realizations of  $\text{tp}(\mathbf{b}/A)$  is not compact, contradicting the fact that  $\mathbf{b} \in \text{acl}(A)$ .  $\square$

#### 4. A SPECIAL CASE: $\aleph_0$ -CATEGORICAL THEORIES

**4.1. Definability and  $\aleph_0$ -Categoricity.** We use our preceding results to characterize  $\aleph_0$ -categorical theories in terms of definability in randomizations.

**Theorem 4.1.1.** *The following are equivalent:*

- (1)  $T$  is  $\aleph_0$ -categorical;
- (2)  $\text{fdcl}_{\mathbb{B}}(A)$  is finite for every finite  $A$ ;
- (3)  $\text{dcl}_{\mathbb{B}}(A)$  is finite for every finite  $A$ ;
- (4)  $\text{fdcl}_{\mathbb{B}}(A) = \text{dcl}_{\mathbb{B}}(A)$  for every finite  $A$ ;
- (5)  $\text{fdcl}(A)$  is finite for every finite  $A$ ;
- (6)  $\text{dcl}(A)$  is finite for every finite  $A$ ;
- (7)  $\text{fdcl}(A) = \text{dcl}(A)$  for every finite  $A$ ;

*Proof.* By the Ryll-Nardzewski Theorem (see [CK], Theorem 2.3.13), (1) is equivalent to

(0) For each  $n$  there are only finitely many formulas in  $n$  variables up to  $T$ -equivalence.

Assume (0) and let  $A \subseteq \widehat{\mathcal{K}}$  be finite. Then (2) holds. Moreover, there are only finitely many functional formulas in  $|A| + 1$  variables, so (5) holds. Then by Corollary 3.3.3, (3), (4), (6), and (7) hold.

Now assume that (0) fails.

*Proof that (2) and (3) fail:* For some  $n$  there are infinitely many formulas in  $n$  variables that are not  $T$ -equivalent. Hence there is an  $n$ -type  $p$  in  $T$  without parameters that is not isolated. So there are formulas  $\varphi_1(\vec{v}), \varphi_2(\vec{v}), \dots$  in  $p$  such that for each  $k > 0$ ,  $T \models \varphi_{k+1} \rightarrow \varphi_k$  but the formula  $\theta_k = \varphi_k \wedge \neg \varphi_{k+1}$  is consistent with  $T$ . The formulas  $\theta_k$  are consistent but pairwise inconsistent. By Fullness, for each  $k > 0$  there exists an  $n$ -tuple  $\vec{\mathbf{b}}_k \in \widehat{\mathcal{K}}^n$  such that  $\llbracket \theta_k(\vec{\mathbf{b}}_k) \rrbracket = \top$ . Since the measured algebra  $(\widehat{\mathcal{B}}, \mu)$  is atomless, there are pairwise disjoint events  $\mathbf{E}_1, \mathbf{E}_2, \dots$  in  $\widehat{\mathcal{B}}$  such that  $\mu(\mathbf{E}_k) = 2^{-k}$  for each  $k > 0$ . By applying Lemma 2.1.8  $k$  times, we see that for each  $k > 0$  there is an  $n$ -tuple  $\vec{\mathbf{a}}_k \in \widehat{\mathcal{K}}^n$  that agrees with  $\vec{\mathbf{b}}_i$  on  $\mathbf{E}_i$  whenever  $0 < i \leq k$ . Whenever  $0 < k \leq j$ , we have  $\mu(\llbracket \vec{\mathbf{a}}_k = \vec{\mathbf{a}}_j \rrbracket) \geq 1 - 2^{-k}$ . So  $\langle \vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \dots \rangle$  is a Cauchy sequence, and by metric completeness the limit  $\vec{\mathbf{a}} = \lim_{k \rightarrow \infty} \vec{\mathbf{a}}_k$  exists in  $\widehat{\mathcal{K}}^n$ . Let  $A = \text{range}(\vec{\mathbf{a}})$ . For each  $k > 0$  we have  $\mathbf{E}_k = \llbracket \vec{\mathbf{a}} = \vec{\mathbf{b}}_k \rrbracket = \llbracket \theta_k(\vec{\mathbf{a}}) \rrbracket$ , so  $\mathbf{E}_k \in \text{fdcl}_{\mathbb{B}}(A)$ . Thus  $\text{fdcl}_{\mathbb{B}}(A)$  is infinite, so (2) fails and (3) fails.

*Proof that (4) fails:* Let  $\mathbf{E}_k$  be as in the preceding paragraph. The set  $\text{fdcl}_{\mathbb{B}}(A)$  is countable. But the closure  $\text{cl}(\text{fdcl}_{\mathbb{B}}(A))$  is uncountable, because for each set  $S \subseteq \mathbb{N} \setminus \{0\}$ , the supremum  $\bigsqcup_{k \in S} \mathbf{E}_k$  belongs to  $\text{cl}(\text{fdcl}_{\mathbb{B}}(A))$ . Thus by Theorem 3.1.2,

$$\text{dcl}_{\mathbb{B}}(A) = \text{cl}(\text{fdcl}_{\mathbb{B}}(A)) \neq \text{fdcl}_{\mathbb{B}}(A),$$

and (4) fails.

*Proof that (5), (6), and (7) fail:* By Corollary 2.1.6, there exist  $\mathbf{c}, \mathbf{d} \in \mathcal{K}$  such that  $\llbracket \mathbf{c} \neq \mathbf{d} \rrbracket = \top$ . Let  $C$  be the finite set  $C = A \cup \{\mathbf{c}, \mathbf{d}\}$ . By Remark 2.2.4, for any event  $D \in \text{fdcl}_{\mathbb{B}}(A)$ , the characteristic function  $1_D$  of  $D$  with respect to  $\mathbf{c}, \mathbf{d}$  is definable over  $C$ . Moreover, we always have  $d_{\mathbb{K}}(1_D, 1_E) = d_{\mathbb{B}}(D, E)$ . It follows that  $\text{fdcl}(C)$  is infinite, so (5) and (6) fail. To show that (7) fails, we take an event  $D \in \text{dcl}_{\mathbb{B}}(A) \setminus \text{fdcl}_{\mathbb{B}}(A)$ . By Theorem 3.1.2 we have  $D \in \text{cl}(\text{fdcl}_{\mathbb{B}}(A))$ . It follows that  $1_D \in \text{cl}(\text{fdcl}(C))$ , so by Theorem 3.3.6,  $1_D \in \text{dcl}(C)$ . Hence  $\text{dcl}(C)$  is uncountable. But  $\text{fdcl}(C)$  is countable, so (7) fails.  $\square$

By the Ryll-Nardzewski Theorem, if  $T$  is  $\aleph_0$ -categorical then for each  $n$ ,  $T$  has finitely many  $n$ -types; so each type  $p$  in the variables  $(u, \vec{v})$  has an *isolating formula*, that is, a formula  $\varphi(u, \vec{v})$  such that  $T \models \varphi(u, \vec{v}) \leftrightarrow \bigwedge p$ .

We now characterize the definable closure of a finite set  $A \subseteq \widehat{\mathcal{K}}$  in the case that  $T$  is  $\aleph_0$ -categorical. Hereafter, when  $A$  is a finite subset of  $\widehat{\mathcal{K}}$ ,  $\vec{\mathbf{a}}$  will denote a finite tuple whose range is  $A$ .

**Corollary 4.1.2.** *Suppose that  $T$  is  $\aleph_0$ -categorical,  $\mathbf{b} \in \widehat{\mathcal{K}}$ , and  $A$  is a finite subset of  $\widehat{\mathcal{K}}$ . Then  $\mathbf{b} \in \text{dcl}(A)$  if and only if:*

- (1)  $\mathbf{b}$  is pointwise definable over  $A$ ;
- (2) for every isolating formula  $\varphi(u, \vec{v})$ , if  $\mu(\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket) > 0$  then

$$\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket = \llbracket (\exists u)\varphi(u, \vec{\mathbf{a}}) \rrbracket.$$

*Proof.* ( $\Rightarrow$ ): Suppose  $\mathbf{b} \in \text{dcl}(A)$ . (1) holds by Lemma 3.2.8. Suppose  $\varphi(u, \vec{v})$  is isolating and  $\mu(\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket) > 0$ . We have  $\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \in \text{fdcl}_{\mathbb{B}}(\{\mathbf{b}\} \cup A)$ , so by Corollary 3.3.5,  $\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \in \text{dcl}_{\mathbb{B}}(A)$ . By Theorem 4.1.1,  $\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \in \text{fdcl}_{\mathbb{B}}(A)$ . We note that  $(\exists u)\varphi(u, \vec{v})$  is an isolating formula, so  $\llbracket (\exists u)\varphi(u, \vec{\mathbf{a}}) \rrbracket$  is an atom of  $\text{fdcl}_{\mathbb{B}}(A)$ . Therefore (2) holds.

( $\Leftarrow$ ): Assume (1) and (2). By (2), for every isolating formula  $\varphi(u, \vec{v})$  such that  $\mu(\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket) > 0$ , we have

$$\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \in \text{fdcl}_{\mathbb{B}}(A).$$

Every formula  $\theta(u, \vec{v})$  is  $T$ -equivalent to a finite disjunction of isolating formulas in the variables  $(u, \vec{v})$ . It follows that  $\text{fdcl}_{\mathbb{B}}(A \cup \{\mathbf{b}\}) \subseteq \text{fdcl}_{\mathbb{B}}(A)$ . Therefore by Corollary 3.3.5,  $\mathbf{b} \in \text{dcl}(A)$ .  $\square$

**Corollary 4.1.3.** *Suppose that  $T$  is  $\aleph_0$ -categorical,  $\mathbf{b} \in \widehat{\mathcal{K}}$ , and  $A$  is a finite subset of  $\widehat{\mathcal{K}}$ . Then  $\mathbf{b} \in \text{dcl}(A)$  if and only if for every isolating formula  $\psi(\vec{v})$  there is a functional formula  $\varphi(u, \vec{v})$  such that  $\llbracket \psi(\vec{\mathbf{a}}) \rrbracket \subseteq \llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket$ .*

*Proof.* ( $\Rightarrow$ ): Suppose  $\mathbf{b} \in \text{dcl}(A)$ . By Theorem 4.1.1,  $\mathbf{b}$  is first order definable over  $\vec{\mathbf{a}}$ , so there is a functional formula  $\varphi(u, \vec{v})$  such that  $\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket = \top$ . Then for every isolating  $\psi(\vec{v})$  we have  $\llbracket \psi(\vec{\mathbf{a}}) \rrbracket \subseteq \llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket$ .

( $\Leftarrow$ ): There is a finite set  $\{\psi_0(\vec{v}), \dots, \psi_k(\vec{v})\}$  that contains exactly one isolating formula for each  $|\vec{\mathbf{a}}|$ -type of  $T$ . By hypothesis, for each  $i \leq k$  there is a functional formula  $\varphi_i(u, \vec{v})$  such that  $\llbracket \psi_i(\vec{\mathbf{a}}) \rrbracket \subseteq \llbracket \varphi_i(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket$ . Since the

formulas  $\psi_i(\vec{v})$  are pairwise inconsistent, the formula  $\bigvee_{i=0}^k (\psi_i(\vec{v}) \wedge \varphi_i(u, \vec{v}))$  is functional, and

$$\llbracket \bigvee_{i=0}^k (\psi_i(\vec{a}) \wedge \varphi_i(\mathbf{b}, \vec{a})) \rrbracket = \top.$$

Hence  $\mathbf{b}$  is first order definable over  $\vec{a}$ , so by Lemma 3.2.3 we have  $\mathbf{b} \in \text{dcl}(A)$ .  $\square$

**4.2. The Theory  $\text{DLO}^R$ .** We will use Corollary 4.1.3 to give a more natural characterization of the definable closure of a finite parameter set in a model of  $\text{DLO}^R$ , where DLO is the theory of dense linear order without endpoints. Note that in DLO, every type in  $(v_1, \dots, v_n)$  has an isolating formula of the form  $\bigwedge_{i=1}^{n-1} u_i \alpha_i u_{i+1}$  where  $\{u_1, \dots, u_n\} = \{v_1, \dots, v_n\}$  and each  $\alpha_i \in \{<, =\}$ . (This formula linearly orders the equality-equivalence classes).

**Corollary 4.2.1.** *Let  $T = \text{DLO}$ ,  $\mathbf{b} \in \widehat{\mathcal{K}}$ , and  $A$  be a finite subset of  $\widehat{\mathcal{K}}$ . Then  $\mathbf{b} \in \text{dcl}(A)$  if and only if for every isolating formula  $\psi(v_1, \dots, v_n)$  there is an  $i \in \{1, \dots, n\}$  such that  $\llbracket \psi(\vec{a}) \rrbracket \sqsubseteq \llbracket \mathbf{b} = \mathbf{a}_i \rrbracket$ .*

*Proof.* For any  $\mathcal{M} \models \text{DLO}$  and parameter set  $A$ , we have  $\text{dcl}^{\mathcal{M}}(A) = A$ . Therefore for every isolating formula  $\psi(v_1, \dots, v_n)$  and functional formula  $\varphi(u, v_1, \dots, v_n)$  there exists  $i \in \{1, \dots, n\}$  such that

$$\text{DLO} \models (\psi(v_1, \dots, v_n) \wedge \varphi(u, v_1, \dots, v_n)) \rightarrow u = v_i.$$

The result now follows from Corollary 4.1.3.  $\square$

In the theory DLO, we define  $\min(u, v)$  and  $\max(u, v)$  in the usual way. For  $\mathbf{a}, \mathbf{b} \in \widehat{\mathcal{K}}$ , we let  $\min(\mathbf{a}, \mathbf{b})$  be the unique element  $\mathbf{e} \in \widehat{\mathcal{K}}$  such that

$$\llbracket \mathbf{e} = \min(\mathbf{a}, \mathbf{b}) \rrbracket = \top,$$

and similarly for  $\max$ . For finite subsets  $A$  of  $\widehat{\mathcal{K}}$ ,  $\min(A)$  and  $\max(A)$  are defined by repeating the two-variable functions  $\min$  and  $\max$  in the natural way.

We next show that in  $\text{DLO}^R$ , the definable closure of a finite set can be characterized as the closure under a “choosing function” of four variables.

**Definition 4.2.2.** In the theory DLO, let  $\ell$  be the function of four variables defined by the condition

$$\ell(u, v, x, y) = x \text{ if } u < v, \text{ and } \ell(u, v, x, y) = y \text{ if not } u < v.$$

For  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathcal{K}$ , let  $\ell(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$  be the unique element  $\mathbf{e} \in \widehat{\mathcal{K}}$  such that  $\llbracket \mathbf{e} = \ell(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \rrbracket = \top$ . Given a set  $A \subseteq \widehat{\mathcal{K}}$ , let  $\text{lcl}(A)$  be the closure of  $A$  under the function  $\ell$ .

Note that in DLO, the function  $\ell$  is definable without parameters. In both DLO and  $\text{DLO}^R$ ,  $\min(u, v) = \ell(u, v, u, v)$ , and  $\max(u, v) = \ell(u, v, v, u)$ .

**Proposition 4.2.3.** *Let  $T = \text{DLO}$ . Then for every finite subset  $A$  of  $\widehat{\mathcal{K}}$ ,  $\text{dcl}(A) = \text{lcl}(A)$ .*

*Proof.* It is clear that  $\text{lcl}(A) \subseteq \text{dcl}(A)$ .

We prove the other inclusion. If  $A$  is empty, the result is trivial, so we assume  $A$  is non-empty. Let  $\mathbf{0} = \min(A)$ ,  $\mathbf{1} = \max(A)$ . We have  $\mathbf{0}, \mathbf{1} \in \text{lcl}(A)$ . Let  $\Omega_0 = \llbracket 0 < 1 \rrbracket$ . Note that  $\Omega \setminus \Omega_0 = \llbracket 0 = 1 \rrbracket$ . If  $\mu(\Omega_0) = 0$ , then  $A$  is a singleton, and we trivially have  $\text{lcl}(A) = \text{dcl}(A) = A$ . We may therefore assume that  $\mu(\Omega_0) > 0$ . To simplify notation we will instead assume that  $\Omega_0 = \Omega$ ; the argument in the general case is similar.

In the following, all characteristic functions are understood to be with respect to  $\mathbf{0}, \mathbf{1}$ . Note that  $\ell(\mathbf{a}, \mathbf{b}, \mathbf{0}, \mathbf{1})$  is the characteristic function of the event  $\llbracket \mathbf{a} < \mathbf{b} \rrbracket$ . If  $\mathbf{d}$  is the characteristic function of an event  $D$  and  $\mathbf{e}$  is the characteristic function of an event  $E$ , then  $\ell(\mathbf{d}, \mathbf{1}, \mathbf{1}, \mathbf{0})$  is the characteristic function of  $\neg D$ ,  $\min(\mathbf{d}, \mathbf{e})$  is the characteristic function of  $D \cap E$ , and  $\max(\mathbf{d}, \mathbf{e})$  is the characteristic function of  $D \sqcup E$ . It follows that for every quantifier-free first order formula  $\varphi(\vec{v})$  of DLO with  $|\vec{v}| = |\vec{\mathbf{a}}|$ , the characteristic function of the event  $\llbracket \varphi(\vec{\mathbf{a}}) \rrbracket$  belongs to  $\text{lcl}(A)$ . Since DLO admits quantifier elimination, the characteristic function of every event that is first order definable over  $A$  belongs to  $\text{lcl}(A)$ . Hence by Theorem 4.1.1, the characteristic function of every event in  $\text{dcl}_{\mathbb{B}}(A)$  belongs to  $\text{lcl}(A)$ . Moreover, for every  $\mathbf{c} \in A$  and event  $D \in \text{dcl}_{\mathbb{B}}(A)$  with characteristic function  $\mathbf{d}$ ,  $\mathbf{c} \upharpoonright D := \ell(\mathbf{d}, \mathbf{1}, \mathbf{0}, \mathbf{c})$  is the element that agrees with  $\mathbf{c}$  on  $D$  and agrees with  $\mathbf{0}$  on the complement of  $D$ , so  $\mathbf{c} \upharpoonright D$  belongs to  $\text{lcl}(A)$ . Let  $\{D_1, \dots, D_n\}$  be the set of atoms of  $\text{dcl}_{\mathbb{B}}(A)$  (which is finite because DLO is  $\aleph_0$ -categorical). By Corollary 4.2.1, every element of  $\text{dcl}(A)$  has the form

$$\max(\mathbf{c}_1 \upharpoonright D_1, \dots, \mathbf{c}_n \upharpoonright D_n)$$

for some  $\mathbf{c}_1, \dots, \mathbf{c}_n \in A$ . Therefore  $\text{dcl}(A) \subseteq \text{lcl}(A)$ .  $\square$

**Example 4.2.4.** In this example we show that the exchange property fails for  $\text{DLO}^R$ , even though it holds for DLO. Thus the exchange property is not preserved under randomizations. Let  $T = \text{DLO}$ . By Fullness, there exist elements  $\mathbf{a}, \mathbf{b} \in \widehat{\mathcal{K}}$  such that  $\max(\mathbf{a}, \mathbf{b}) \notin \{\mathbf{a}, \mathbf{b}\}$ . Let  $\mathbf{c} = \max(\mathbf{a}, \mathbf{b})$ ,  $\mathbf{d} = \min(\mathbf{a}, \mathbf{b})$ . It is easy to check that

$$\text{dcl}(\{\mathbf{a}, \mathbf{b}\}) = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}, \quad \text{dcl}(\{\mathbf{a}, \mathbf{c}\}) = \{\mathbf{a}, \mathbf{c}\}, \quad \text{dcl}(\{\mathbf{a}\}) = \{\mathbf{a}\}.$$

Thus  $\mathbf{c} \in \text{dcl}(\{\mathbf{a}, \mathbf{b}\}) \setminus \text{dcl}(\{\mathbf{a}\})$  but  $\mathbf{b} \notin \text{dcl}(\{\mathbf{a}, \mathbf{c}\})$ .

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