

DEFINABLE CLOSURE IN RANDOMIZATIONS

URI ANDREWS, ISAAC GOLDBRING, AND H. JEROME KEISLER

ABSTRACT. The randomization of a complete first order theory T is the complete continuous theory T^R with two sorts, a sort for random elements of models of T , and a sort for events in an underlying probability space. We give necessary and sufficient conditions for an element to be definable over a set of parameters in a model of T^R .

1. INTRODUCTION

A randomization of a first order structure \mathcal{M} , as introduced by Keisler [Kei1] and formalized as a metric structure by Ben Yaacov and Keisler [BK], is a continuous structure \mathcal{N} with two sorts, a sort for random elements of \mathcal{M} , and a sort for events in an underlying atomless probability space. Given a complete first order theory T , the theory T^R of randomizations of models of T forms a complete theory in continuous logic, which is called the randomization of T . In a model \mathcal{N} of T^R , for each n -tuple \vec{a} of random elements and each first order formula $\varphi(\vec{v})$, the set of points in the underlying probability space where $\varphi(\vec{a})$ is true is an event denoted by $\llbracket \varphi(\vec{a}) \rrbracket$.

In a first order structure \mathcal{M} , an element b is *definable over* a set A of elements of \mathcal{M} (called parameters) if there is a tuple \vec{a} in A and a formula $\varphi(u, \vec{a})$ such that

$$\mathcal{M} \models (\forall u)(\varphi(u, \vec{a}) \leftrightarrow u = b).$$

In a general metric structure \mathcal{N} , an element b is said to be *definable over* a set of parameters A if there is a sequence of tuples \vec{a}_n in A and continuous formulas $\Phi_n(x, \vec{a}_n)$ whose truth values converge uniformly to the distance from x to b . In this paper we give necessary and sufficient conditions for definability in a model of the randomization theory T^R . These conditions can be stated in terms of sequences of first order formulas. The results in this paper will be applied in a forthcoming paper about independence relations in randomizations.

In Theorem 3.1.2, we show that an event E is definable over a set A of parameters if and only if it is the limit of a sequence of events of the form $\llbracket \varphi_n(\vec{a}_n) \rrbracket$, where each φ_n is a first order formula and each \vec{a}_n is a tuple from A .

In Theorem 3.3.6, we show that a random element b is definable over a set A of parameters if and only if b is the limit of a sequence of random elements

2010 *Mathematics Subject Classification*. Primary 03C40. Secondary 03B48, 03B50, 03C35.

b_n such that for each n ,

$$\llbracket (\forall u)(\varphi_n(u, \vec{a}_n) \leftrightarrow u = b_n) \rrbracket$$

has probability one for some first order formula $\varphi_n(u, \vec{v})$ and a tuple \vec{a}_n from A . In Section 4 we give some consequences in the special case that the underlying first order theory T is \aleph_0 -categorical.

Continuous model theory in its current form is developed in the papers [BBHU] and [BU]. The papers [Go1], [Go2], [Go3] deal with definability questions in metric structures. Randomizations of models are treated in [AK], [Be], [BK], [EG], [GL], [Ke1], and [Ke2].

2. PRELIMINARIES

We refer to [BBHU] and [BU] for background in continuous model theory, and follow the notation of [BK]. We assume familiarity with the basic notions about continuous model theory as developed in [BBHU], including the notions of a theory, structure, pre-structure, model of a theory, elementary extension, isomorphism, and κ -saturated structure. In particular, the universe of a pre-structure is a pseudo-metric space, the universe of a structure is a complete metric space, and every pre-structure has a unique completion. In continuous logic, formulas have truth values in the unit interval $[0, 1]$ with 0 meaning true, the connectives are continuous functions from $[0, 1]^n$ into $[0, 1]$, and the quantifiers are sup and inf. A *tuple* is a finite sequence, and $A^{<\mathbb{N}}$ is the set of all tuples of elements of A .

2.1. The theory T^R . We assume throughout that L is a finite or countable first order signature, and that T is a complete theory for L whose models have at least two elements.

The *randomization signature* L^R is the two-sorted continuous signature with sorts \mathbb{K} (for random elements) and \mathbb{B} (for events), an n -ary function symbol $\llbracket \varphi(\cdot) \rrbracket$ of sort $\mathbb{K}^n \rightarrow \mathbb{B}$ for each first order formula φ of L with n free variables, a $[0, 1]$ -valued unary predicate symbol μ of sort \mathbb{B} for probability, and the Boolean operations $\top, \perp, \sqcap, \sqcup, \neg$ of sort \mathbb{B} . The signature L^R also has distance predicates $d_{\mathbb{B}}$ of sort \mathbb{B} and $d_{\mathbb{K}}$ of sort \mathbb{K} . In L^R , we use B, C, \dots for variables or parameters of sort \mathbb{B} . $B \dot{=} C$ means $d_{\mathbb{B}}(B, C) = 0$, and $B \sqsubseteq C$ means $B \dot{=} B \sqcap C$.

A pre-structure for T^R will be a pair $\mathcal{P} = (\mathcal{K}, \mathcal{B})$ where \mathcal{K} is the part of sort \mathbb{K} and \mathcal{B} is the part of sort \mathbb{B} . The *reduction* of \mathcal{P} is the pre-structure $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$ obtained from \mathcal{P} by identifying elements at distance zero, and the associated mapping from \mathcal{P} onto \mathcal{N} is called the *reduction map*. The *completion* of \mathcal{P} is the structure obtained by completing the metrics in the reduction of \mathcal{P} . A pre-structure \mathcal{P} is called *pre-complete* if the reduction of \mathcal{P} is already the completion of \mathcal{P} .

In [BK], the randomization theory T^R is defined by listing a set of axioms. We will not repeat these axioms here, because it is simpler to give the following model-theoretic characterization of T^R .

Definition 2.1.1. Given a model \mathcal{M} of T , a *nice randomization* of \mathcal{M} is a pre-complete structure $(\mathcal{K}, \mathcal{B})$ for L^R equipped with an atomless probability space $(\Omega, \mathcal{B}, \mu)$ such that:

- (1) \mathcal{B} is a σ -algebra with $\top, \perp, \sqcap, \sqcup, \neg$ interpreted by $\Omega, \emptyset, \cap, \cup, \setminus$.
- (2) \mathcal{K} is a set of functions $a: \Omega \rightarrow M$.
- (3) For each formula $\psi(\vec{x})$ of L and tuple \vec{a} in \mathcal{K} , we have

$$\llbracket \psi(\vec{a}) \rrbracket = \{\omega \in \Omega : \mathcal{M} \models \psi(\vec{a}(\omega))\} \in \mathcal{B}.$$

- (4) \mathcal{B} is equal to the set of all events $\llbracket \psi(\vec{a}) \rrbracket$ where $\psi(\vec{v})$ is a formula of L and \vec{a} is a tuple in \mathcal{K} .
- (5) For each formula $\theta(u, \vec{v})$ of L and tuple \vec{b} in \mathcal{K} , there exists $a \in \mathcal{K}$ such that

$$\llbracket \theta(a, \vec{b}) \rrbracket = \llbracket (\exists u \theta)(\vec{b}) \rrbracket.$$

- (6) On \mathcal{K} , the distance predicate $d_{\mathbb{K}}$ defines the pseudo-metric

$$d_{\mathbb{K}}(a, b) = \mu[a \neq b].$$

- (7) On \mathcal{B} , the distance predicate $d_{\mathbb{B}}$ defines the pseudo-metric

$$d_{\mathbb{B}}(\mathbf{B}, \mathbf{C}) = \mu(\mathbf{B} \Delta \mathbf{C}).$$

Definition 2.1.2. For each first order theory T , the *randomization theory* T^R is the set of sentences that are true in all nice randomizations of models of T .

It follows that for each first order sentence φ , if $T \models \varphi$ then $T^R \models \llbracket \varphi \rrbracket \doteq \top$. The following basic facts are from [BK], Theorem 2.1 and Proposition 2.2, Example 3.4 (ii), Proposition 2.7, and Theorem 2.9.

Fact 2.1.3. For every complete first order theory T , the randomization theory T^R is complete.

Fact 2.1.4. Every model \mathcal{M} of T has nice randomizations.

Fact 2.1.5. (Fullness) Every pre-complete model $\mathcal{P} = (\mathcal{K}, \mathcal{B})$ of T^R has perfect witnesses, i.e.,

- (1) For each first order formula $\theta(u, \vec{v})$ and each \vec{b} in \mathcal{K}^n there exists $a \in \mathcal{K}$ such that

$$\llbracket \theta(a, \vec{b}) \rrbracket \doteq \llbracket (\exists u \theta)(\vec{b}) \rrbracket;$$

- (2) For each $\mathbf{B} \in \mathcal{B}$ there exist $a, b \in \mathcal{K}$ such that $\mathbf{B} \doteq \llbracket a = b \rrbracket$.

Corollary 2.1.6. Every model \mathcal{N} of T^R has a pair of elements c, d such that $\llbracket c \neq d \rrbracket = \top$.

Proof. Every model of T has at least two elements, so $T \models (\exists u)(\exists v)u \neq v$. The result follows by applying Fullness twice. \square

Fact 2.1.7. (Strong quantifier elimination) Every formula Φ in the continuous language L^R is T^R -equivalent to a formula with the same free variables and no quantifiers of sort \mathbb{K} or \mathbb{B} .

Lemma 2.1.8. *Let $\mathcal{P} = (\mathcal{K}, \mathcal{B})$ be a pre-complete model of T^R and let $a, b \in \mathcal{K}$ and $B \in \mathcal{B}$. Then there is an element $c \in \mathcal{K}$ that agrees with a on B and agrees with b on $\neg B$, that is, $B \sqsubseteq \llbracket c = a \rrbracket$ and $(\neg B) \sqsubseteq \llbracket c = b \rrbracket$.*

Definition 2.1.9. In Lemma 2.1.8, we will call c a *characteristic function* of B with respect to a, b .

Note that the distance between any two characteristic functions of an event B with respect to elements a, b is zero. In particular, in a model of T^R , the characteristic function is unique.

Proof of Lemma 2.1.8. By Fact 2.1.5 (2), there exist $d, e \in \mathcal{K}$ such that $B \doteq \llbracket d = e \rrbracket$. The first order sentence

$$(\forall u)(\forall v)(\forall x)(\forall y)(\exists z)[(x = y \rightarrow z = u) \wedge (x \neq y \rightarrow z = v)]$$

is logically valid, so we must have

$$\llbracket (\exists z)[(d = e \rightarrow z = a) \wedge (d \neq e \rightarrow z = b)] \rrbracket \doteq \top.$$

By Fact 2.1.5 (1) there exists $c \in \mathcal{K}$ such that

$$\llbracket d = e \rightarrow c = a \rrbracket \doteq \top, \quad \llbracket d \neq e \rightarrow c = b \rrbracket \doteq \top,$$

so $\llbracket d = e \rrbracket \sqsubseteq \llbracket c = a \rrbracket$ and $\llbracket d \neq e \rrbracket \sqsubseteq \llbracket c = b \rrbracket$. \square

We will need the following result, which is a consequence of Theorem 3.11 of [Be]. Since the setting in [Be] is quite different from the present paper, we give a direct proof here.

Proposition 2.1.10. *Every model of T^R is isomorphic to the reduction of a nice randomization of a model of T .*

Proof. Let $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$ be a model of T^R of cardinality κ . Let Ω be the Stone space of the Boolean algebra $\widehat{\mathcal{B}} = (\widehat{\mathcal{B}}, \top, \perp, \sqcap, \sqcup, \neg)$. Thus Ω is a compact topological space, the points of Ω are ultrafilters, we may identify $\widehat{\mathcal{B}}$ with the Boolean algebra of clopen sets of Ω , and $\mu^{\mathcal{N}}$ is a finitely additive probability measure on $\widehat{\mathcal{B}}$.

We next show that μ is σ -additive on $\widehat{\mathcal{B}}$. To do this, we assume that $A_0 \supseteq A_1 \supseteq \dots$ in $\widehat{\mathcal{B}}$ and $C = \bigcap_n A_n \in \widehat{\mathcal{B}}$, and prove that $\mu(C) = \lim_{n \rightarrow \infty} \mu(A_n)$. Indeed, the family $\{C \cup (\Omega \setminus A_n) : n \in \mathbb{N}\}$ is an open covering of Ω , so by the topological compactness of Ω , we have $\Omega = \bigcup_{k=0}^n (C \cup (\Omega \setminus A_k))$ for some $n \in \mathbb{N}$. Then $C = A_n$, so $\mu(C) = \mu(A_n) = \lim_{n \rightarrow \infty} \mu(A_n)$.

By the Caratheodory theorem, there is a complete probability space $(\Omega, \mathcal{B}, \mu)$ such that $\mathcal{B} \supseteq \widehat{\mathcal{B}}$, μ agrees with $\mu^{\mathcal{N}}$ on $\widehat{\mathcal{B}}$, and for each $B \in \mathcal{B}$ and $m > 0$ there is a countable sequence $A_{m0} \subseteq A_{m1} \subseteq \dots$ in $\widehat{\mathcal{B}}$ such that

$$(2.1) \quad B \subseteq \bigcup_n A_{mn} \text{ and } \mu \left(\bigcup_n A_{mn} \right) \leq \mu(B) + 1/m.$$

Note that since the probability space $(\Omega, \mathcal{B}, \mu)$ is complete, every subset of Ω that contains a set in \mathcal{B} of measure one also belongs to \mathcal{B} and has measure one.

We claim that for each $\mathbf{B} \in \mathcal{B}$ there is a unique event $f(\mathbf{B}) \in \widehat{\mathcal{B}}$ such that $\mu(f(\mathbf{B})\Delta\mathbf{B}) = 0$. The uniqueness of $f(\mathbf{B})$ follows from the fact that the distance function $d_{\mathbb{B}}(\mathbf{C}, \mathbf{D}) = \mu(\mathbf{C}\Delta\mathbf{D})$ is a metric on $\widehat{\mathcal{B}}$. To show the existence of $f(\mathbf{B})$, for each $m > 0$ let $\mathbf{A}_{m0} \subseteq \mathbf{A}_{m1} \subseteq \dots$ be as in (2.1). Note that $(\mathbf{A}_{m0}, \mathbf{A}_{m1}, \dots)$ is a Cauchy sequence of events in the model \mathcal{N} , so there is an event $\mathbf{C}_m \in \widehat{\mathcal{B}}$ such that $\mathbf{C}_m = \lim_{n \rightarrow \infty} \mathbf{A}_{mn}$. Hence $\lim_{n \rightarrow \infty} \mu(\mathbf{A}_{mn}\Delta\mathbf{C}_m) = 0$, so $\mu((\bigcup_n \mathbf{A}_{mn})\Delta\mathbf{C}_m) = 0$. Then $(\mathbf{C}_1, \mathbf{C}_2, \dots)$ is a Cauchy sequence, so there is an event $f(\mathbf{B}) = \lim_{m \rightarrow \infty} \mathbf{C}_m$ in $\widehat{\mathcal{B}}$ with $\mu(f(\mathbf{B})\Delta\mathbf{B}) = 0$.

We make some observations about the mapping $f: \mathcal{B} \rightarrow \widehat{\mathcal{B}}$. If $\mathbf{B}, \mathbf{C} \in \mathcal{B}$ and $d_{\mathbb{B}}(\mathbf{B}, \mathbf{C}) = 0$, then $f(\mathbf{B}) = f(\mathbf{C})$. For each $\mathbf{B}, \mathbf{C} \in \mathcal{B}$, we have

$$\begin{aligned} f(\mathbf{B} \cup \mathbf{C}) &= f(\mathbf{B}) \cup f(\mathbf{C}), & f(\mathbf{B} \cap \mathbf{C}) &= f(\mathbf{B}) \cap f(\mathbf{C}), \\ \Omega \setminus f(\mathbf{B}) &= f(\Omega \setminus \mathbf{B}), & \mu(\mathbf{B}) &= \mu(f(\mathbf{B})). \end{aligned}$$

Moreover, the mapping f sends \mathcal{B} onto $\widehat{\mathcal{B}}$, because if $\mathbf{C} \in \widehat{\mathcal{B}}$ then $\mathbf{C} \in \mathcal{B}$ and $f(\mathbf{C}) = \mathbf{C}$. Therefore the mapping \hat{f} that sends the equivalence class of each $\mathbf{B} \in \mathcal{B}$ under $d_{\mathbb{B}}$ to $f(\mathbf{B})$ is well defined and is an isomorphism from the reduction of the pre-structure $(\mathcal{B}, \sqcup, \sqcap, \neg, \top, \perp, \mu)$ onto the measured algebra $(\widehat{\mathcal{B}}, \sqcup, \sqcap, \neg, \top, \perp, \mu)$.

A model \mathcal{M} of T is κ^+ -universal if every model of T of cardinality $\leq \kappa$ is elementarily embeddable in \mathcal{M} . By Theorem 5.1.12 in [CK], every κ -saturated model of T is κ^+ -universal, so κ^+ -universal models of T exist. We now assume that \mathcal{M} is a κ^+ -universal model of T , and prove that \mathcal{N} is isomorphic to the reduction of a nice randomization of \mathcal{M} with the underlying probability space $(\Omega, \mathcal{B}, \mu)$.

In the following paragraphs, we will use boldface letters $\mathbf{b}, \mathbf{d}, \dots$ for elements of $\widehat{\mathcal{K}}$. Let $L_{\widehat{\mathcal{K}}}$ be the first order signature formed by adding a constant symbol for each element $\mathbf{b} \in \widehat{\mathcal{K}}$. For each $\omega \in \Omega$, the set of $L_{\widehat{\mathcal{K}}}$ -sentences

$$U(\omega) = \{\psi(\vec{\mathbf{b}}) : \omega \in \llbracket \psi(\vec{\mathbf{b}}) \rrbracket\}$$

is consistent with T and has cardinality $\leq \kappa$. By the Compactness and Löwenheim-Skolem theorems, each $U(\omega)$ has a model $(\mathcal{M}_\omega, \mathbf{b}_\omega)_{\mathbf{b} \in \widehat{\mathcal{K}}}$ of cardinality $\leq \kappa$. Since \mathcal{M} is κ^+ -universal, for each $\omega \in \Omega$ we may choose an elementary embedding $h_\omega: \mathcal{M}_\omega \prec \mathcal{M}$. Then $(\mathcal{M}, h_\omega(\mathbf{b}_\omega))_{\mathbf{b} \in \widehat{\mathcal{K}}} \models U(\omega)$ for every $\omega \in \Omega$. It follows that for each formula $\psi(\vec{v})$ of L and each tuple $\vec{\mathbf{b}} \in \widehat{\mathcal{K}}^{<\mathbb{N}}$,

$$\llbracket \psi(\vec{\mathbf{b}}) \rrbracket = \{\omega \in \Omega : \mathcal{M}_\omega \models \psi(\vec{\mathbf{b}}_\omega)\} = \{\omega \in \Omega : \mathcal{M} \models \psi(h_\omega(\vec{\mathbf{b}}_\omega))\} \in \widehat{\mathcal{B}}.$$

For each formula $\psi(\vec{v})$ of L and tuple \vec{c} of functions in M^Ω , define

$$\llbracket \psi(\vec{c}) \rrbracket := \{\omega \in \Omega : \mathcal{M} \models \psi(\vec{c}(\omega))\}.$$

Let \mathcal{K} be the set of all functions $a: \Omega \rightarrow M$ such that for some element $\mathbf{b} \in \widehat{\mathcal{K}}$, we have

$$\mu(\{\omega \in \Omega: a(\omega) = h_\omega(\mathbf{b}_\omega)\}) = 1.$$

We claim that for each $a \in \mathcal{K}$ there is a unique element $f(a) \in \widehat{\mathcal{K}}$ such that

$$\mu(\{\omega \in \Omega: a(\omega) = h_\omega(f(a)_\omega)\}) = 1.$$

The existence of $f(a)$ is guaranteed by the definition of \mathcal{K} . To prove uniqueness, suppose $\mathbf{b}, \mathbf{d} \in \widehat{\mathcal{K}}$ and

$$\mu(\{\omega \in \Omega: a(\omega) = h_\omega(\mathbf{b}_\omega)\}) = \mu(\{\omega \in \Omega: a(\omega) = h_\omega(\mathbf{d}_\omega)\}) = 1.$$

Then

$$\mu(\{\omega \in \Omega: h_\omega(\mathbf{b}_\omega) = h_\omega(\mathbf{d}_\omega)\}) = 1,$$

so

$$\mu(\llbracket \mathbf{b} = \mathbf{d} \rrbracket) = \mu(\{\omega \in \Omega: \mathbf{b}_\omega = \mathbf{d}_\omega\}) = 1,$$

and hence $d_{\mathbb{K}}(\mathbf{b}, \mathbf{d}) = 0$. Since $d_{\mathbb{K}}$ is a metric on $\widehat{\mathcal{K}}$, it follows that $\mathbf{b} = \mathbf{d}$.

We now make some observations about the mapping $f: \mathcal{K} \rightarrow \widehat{\mathcal{K}}$. This mapping sends \mathcal{K} onto $\widehat{\mathcal{K}}$, because for each $\mathbf{b} \in \widehat{\mathcal{K}}$, we have $f(a) = \mathbf{b}$ where a is the element of \mathcal{K} such that $a(\omega) = h_\omega(\mathbf{b}_\omega)$ for all $\omega \in \Omega$. Suppose $\vec{c} \in \mathcal{K}^{<\mathbb{N}}$ and $\vec{\mathbf{d}} = f(\vec{c})$. We have $\vec{\mathbf{d}} \in \widehat{\mathcal{K}}^{<\mathbb{N}}$ and

$$\llbracket \psi(\vec{\mathbf{d}}) \rrbracket = \{\omega \in \Omega: \mathcal{M} \models \psi(h_\omega(\vec{\mathbf{d}}_\omega))\} \doteq \{\omega \in \Omega: \mathcal{M} \models \psi(\vec{c}(\omega))\} = \llbracket \psi(\vec{c}) \rrbracket.$$

Since the probability space $(\Omega, \mathcal{B}, \mu)$ is complete, $\llbracket \psi(\vec{\mathbf{d}}) \rrbracket \in \widehat{\mathcal{B}} \subseteq \mathcal{B}$, and $\llbracket \psi(\vec{\mathbf{d}}) \rrbracket \doteq \llbracket \psi(\vec{c}) \rrbracket$, we have $\llbracket \psi(\vec{c}) \rrbracket \in \mathcal{B}$ and $\llbracket \psi(\vec{\mathbf{d}}) \rrbracket = f(\llbracket \psi(\vec{c}) \rrbracket)$. Therefore, if $a, c \in \mathcal{K}$ and $d_{\mathbb{K}}(a, c) = 0$, then $d_{\mathbb{K}}(f(a), f(c)) = 0$, and hence $f(a) = f(c)$. This shows that $\mathcal{P} = (\mathcal{K}, \mathcal{B})$ is a well-defined pre-complete structure for L^R , and that the mapping \widehat{f} that sends the equivalence class of each $\mathbf{B} \in \mathcal{B}$ to $f(\mathbf{B})$, and the equivalence class of each $a \in \mathcal{K}$ to $f(a)$, is an isomorphism from the reduction of \mathcal{P} to \mathcal{N} .

It remains to show that \mathcal{P} is a nice randomization of \mathcal{M} . It is clear that \mathcal{P} satisfies conditions (1)-(3) in Definition 2.1.1.

Proof of (4): We have already shown that $\llbracket \psi(\vec{c}) \rrbracket \in \mathcal{B}$ for each formula $\psi(\vec{v})$ of L and each tuple \vec{c} in \mathcal{K} . For the other direction, let $\mathbf{B} \in \mathcal{B}$. By Corollary 2.1.6, there exist $a, e \in \mathcal{K}$ such that $\llbracket a \neq e \rrbracket \doteq \Omega$. We may choose a function $b \in M^\Omega$ such that $b(\omega) = e(\omega)$ whenever $a(\omega) \neq e(\omega)$, and $b(\omega) \neq a(\omega)$ for all $\omega \in \Omega$. Then $b \in \mathcal{K}$ and $\llbracket a \neq b \rrbracket = \Omega$. By Lemma 2.1.8, there exists $c \in \mathcal{K}$ which is a characteristic function of \mathbf{B} with respect to a, b . Then $\llbracket c = a \rrbracket \doteq \mathbf{B}$. Let $d \in M^\Omega$ be the function such that $d(\omega) = a(\omega)$ for $\omega \in \mathbf{B}$, and $d(\omega) = b(\omega)$ for $\omega \in \neg\mathbf{B}$. Then $\mu(\llbracket c = d \rrbracket) = 1$, so $d \in \mathcal{K}$, and $\llbracket a = d \rrbracket = \mathbf{B}$. Thus (4) holds with ψ being the sentence $a = d$.

Proof of (5): Consider a formula $\theta(u, \vec{v})$ of L and a tuple \vec{b} in \mathcal{K} . By Fullness, there exists $c \in \mathcal{K}$ such that

$$\llbracket \theta(c, \vec{b}) \rrbracket \doteq \llbracket (\exists u)\theta(u, \vec{b}) \rrbracket.$$

We may choose a function $a \in M^\Omega$ such that for all $\omega \in \Omega$,

$$\mathcal{M} \models [\theta(c(\omega), \vec{b}(\omega)) \leftrightarrow (\exists u)\theta(u, \vec{b})] \text{ implies } a(\omega) = c(\omega),$$

and

$$\mathcal{M} \models [(\exists u)\theta(u, \vec{b}(\omega)) \rightarrow \theta(a(\omega), \vec{b}(\omega))].$$

Then $\mu(\llbracket a = c \rrbracket) = 1$, so $a \in \mathcal{K}$ and

$$\llbracket \theta(a, \vec{b}) \rrbracket = \llbracket (\exists u)\theta(u, \vec{b}) \rrbracket,$$

as required.

Proof of (6) and (7): By Fact 2.1.4, the properties

$$(\forall x)(\forall y)d_{\mathbb{K}}(x, y) = \mu(\llbracket x \neq y \rrbracket), \quad (\forall \mathbf{U})(\forall \mathbf{V})d_{\mathbb{B}}(\mathbf{U}, \mathbf{V}) = \mu(\mathbf{U} \Delta \mathbf{V})$$

hold in some model of T^R . By Fact 2.1.3, these properties hold in all models of T^R , and thus in \mathcal{N} . Therefore (6) and (7) hold for \mathcal{P} . \square

2.2. Types and Definability. For a first order structure \mathcal{M} and a set A of elements of \mathcal{M} , \mathcal{M}_A denotes the structure formed by adding a new constant symbol to \mathcal{M} for each $a \in A$. The *type realized by a tuple \vec{b}* over the parameter set A in \mathcal{M} is the set $\text{tp}^{\mathcal{M}}(\vec{b}/A)$ of formulas $\varphi(\vec{u}, \vec{a})$ with $\vec{a} \in A^{<\mathbb{N}}$ satisfied by \vec{b} in \mathcal{M}_A . We call $\text{tp}^{\mathcal{M}}(\vec{b}/A)$ an *n-type* if $n = |\vec{b}|$.

In the following, let \mathcal{N} be a continuous structure and let A be a set of elements of \mathcal{N} . \mathcal{N}_A denotes the structure formed by adding a new constant symbol to \mathcal{N} for each $a \in A$. The *type $\text{tp}^{\mathcal{N}}(\vec{b}/A)$ realized by \vec{b}* over the parameter set A in \mathcal{N} is the function p from formulas to $[0, 1]$ such that for each formula $\Phi(\vec{x}, \vec{a})$ with $\vec{a} \in A^{<\mathbb{N}}$, we have $\Phi(\vec{x}, \vec{a})^p = \Phi(\vec{b}, \vec{a})^{\mathcal{N}}$.

We now recall the notions of definable element and algebraic element from [BBHU]. An element b is *definable over A* in \mathcal{N} , in symbols $b \in \text{dcl}^{\mathcal{N}}(A)$, if there is a sequence of formulas $\langle \Phi_k(x, \vec{a}_k) \rangle$ with $\vec{a}_k \in A^{<\mathbb{N}}$ such that the sequence of functions $\langle \Phi_k(x, \vec{a}_k)^{\mathcal{N}} \rangle$ converges uniformly in x to the distance function $d(x, b)^{\mathcal{N}}$ of the corresponding sort. b is *algebraic over A* in \mathcal{N} , in symbols $b \in \text{acl}^{\mathcal{N}}(A)$, if there is a compact set C and a sequence of formulas $\langle \Phi_k(x, \vec{a}_k) \rangle$ with $\vec{a}_k \in A^{<\mathbb{N}}$ such that $b \in C$ and the sequence of functions $\langle \Phi_k(x, \vec{a}_k)^{\mathcal{N}} \rangle$ converges uniformly in x to the distance function $d(x, C)^{\mathcal{N}}$ of the corresponding sort.

If the structure \mathcal{N} is clear from the context, we will sometimes drop the superscript and write $\text{tp}, \text{dcl}, \text{acl}$ instead of $\text{tp}^{\mathcal{N}}, \text{dcl}^{\mathcal{N}}, \text{acl}^{\mathcal{N}}$.

Fact 2.2.1. ([BBHU], Exercises 10.7 and 10.10) For each element b of \mathcal{N} , the following are equivalent, where $p = \text{tp}^{\mathcal{N}}(b/A)$:

- (1) b is definable over A in \mathcal{N} ;
- (2) in each model $\mathcal{N}' \succ \mathcal{N}$, b is the a unique element that realizes p over A ;
- (3) b is definable over some countable subset of A in \mathcal{N} .

Fact 2.2.2. ([BBHU], Exercise 10.8 and 10.11) For each element b of \mathcal{N} , the following are equivalent, where $p = \text{tp}^{\mathcal{N}}(b/A)$:

- (1) b is algebraic over A in \mathcal{N} ;
- (2) in each model $\mathcal{N}' \succ \mathcal{N}$, the set of elements b that realize p over A in \mathcal{N}' is compact.
- (3) b is algebraic over some countable subset of A in \mathcal{N} .

Fact 2.2.3. (Definable Closure, Exercises 10.10 and 10.11 in [BBHU])

- (1) If $A \subseteq \mathcal{N}$ then $\text{dcl}(A) = \text{dcl}(\text{dcl}(A))$ and $\text{acl}(A) = \text{acl}(\text{acl}(A))$.
- (2) If A is a dense subset of B and $B \subseteq \mathcal{N}$, then $\text{dcl}(A) = \text{dcl}(B)$ and $\text{acl}(A) = \text{acl}(B)$.

It follows that for any $A \subseteq \mathcal{N}$, $\text{dcl}(A)$ and $\text{acl}(A)$ are closed with respect to the metric in \mathcal{N} .

We now turn to the case where \mathcal{N} is a model of T^R . In that case, a set of elements of \mathcal{N} may contain elements of both sorts \mathbb{K}, \mathbb{B} . But as we will now explain, we need only consider definability over sets of parameters of sort \mathbb{K} .

Remark 2.2.4. Let $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$ be a model of T^R . Since every model of T has at least two elements, \mathcal{N} has a pair of elements a, b of sort \mathbb{K} such that $\mathcal{N} \models \llbracket a = b \rrbracket = \perp$. For each event $\mathbb{D} \in \widehat{\mathcal{B}}$, let $1_{\mathbb{D}}$ be the characteristic function of \mathbb{D} with respect to a, b . Then in the model \mathcal{N} , \mathbb{D} is definable over $\{a, b, 1_{\mathbb{D}}\}$, and $1_{\mathbb{D}}$ is definable over $\{a, b, \mathbb{D}\}$.

Proof. By Fact 2.2.1. □

In view of Remark 2.2.4 and Fact 2.2.3, if C is a set of parameters in \mathcal{N} of both sorts, and there are elements $a, b \in C$ such that $\mathcal{N} \models \llbracket a = b \rrbracket = \perp$, then an element of either sort is definable over C if and only if it is definable over the set of parameters of sort \mathbb{K} obtained by replacing each element of C of sort \mathbb{B} by its characteristic function with respect to a, b . For this reason, in a model \mathcal{N} of T^R we will only consider definability over sets of parameters of sort \mathbb{K} . We write $\text{dcl}_{\mathbb{B}}(A)$ for the set of elements of sort \mathbb{B} that are definable over A in \mathcal{N} , and write $\text{dcl}(A)$ for the set of elements of sort \mathbb{K} that are definable over A in \mathcal{N} . Similarly for $\text{acl}_{\mathbb{B}}(A)$ and $\text{acl}(A)$.

2.3. Conventions and Notation. We will assume hereafter that $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$ is a model of T^R , $\mathcal{P} = (\mathcal{K}, \mathcal{B})$ is a nice randomization of a model $\mathcal{M} \models T$ with probability space $(\Omega, \mathcal{B}, \mu)$, and \mathcal{N} is the reduction of \mathcal{P} . The existence of \mathcal{P} is guaranteed by Proposition 2.1.10.

We will use boldfaced letters $\mathbf{a}, \mathbf{b}, \dots$ for elements of $\widehat{\mathcal{K}}$. For each element $\mathbf{a} \in \widehat{\mathcal{K}}$, we will choose once and for all an element $a \in \mathcal{K}$ such that the image of a under the reduction map is \mathbf{a} . It follows that for each first order formula $\varphi(\vec{v})$, $\llbracket \varphi(\vec{\mathbf{a}}) \rrbracket$ is the image of $\llbracket \varphi(\vec{a}) \rrbracket$ under the reduction map. For any countable set $A \subseteq \widehat{\mathcal{K}}$ and each $\omega \in \Omega$, we define

$$A(\omega) = \{a(\omega) : \mathbf{a} \in A\}.$$

When $A \subseteq \widehat{\mathcal{K}}$, $\text{cl}(A)$ denotes the closure of A in the metric $d_{\mathbb{K}}$. When $B \subseteq \widehat{\mathcal{B}}$, $\text{cl}(B)$ denotes the closure of B in the metric $d_{\mathbb{B}}$, and $\sigma(B)$ denotes the smallest σ -subalgebra of $\widehat{\mathcal{B}}$ containing B .

3. RANDOMIZATIONS OF ARBITRARY THEORIES

3.1. Definability in Sort \mathbb{B} . We characterize the set of elements of $\widehat{\mathcal{B}}$ that are definable in \mathcal{N} over a set of parameters $A \subseteq \widehat{\mathcal{K}}$.

Definition 3.1.1. For each $A \subseteq \widehat{\mathcal{K}}$, we say that an event E is *first order definable* over A , in symbols $E \in \text{fdcl}_{\mathbb{B}}(A)$, if $E = \llbracket \varphi(\vec{a}) \rrbracket$ for some first order formula $\varphi(\vec{v})$ and tuple \vec{a} in $A^{<\mathbb{N}}$.

Theorem 3.1.2. For each $A \subseteq \widehat{\mathcal{K}}$, $\text{dcl}_{\mathbb{B}}(A) = \text{cl}(\text{fdcl}_{\mathbb{B}}(A)) = \sigma(\text{fdcl}_{\mathbb{B}}(A))$.

Proof. By quantifier elimination (Fact 2.1.7), in any elementary extension $\mathcal{N}' \succ \mathcal{N}$, two events have the same type over A if and only if they have the same type over $\text{fdcl}_{\mathbb{B}}(A)$. Then by Fact 2.2.1, $\text{dcl}_{\mathbb{B}}(A) = \text{dcl}_{\mathbb{B}}(\text{fdcl}_{\mathbb{B}}(A))$. Moreover, $\text{dcl}_{\mathbb{B}}(\text{fdcl}_{\mathbb{B}}(A))$ is equal to the definable closure of $\text{fdcl}_{\mathbb{B}}(A)$ in the pure measured algebra $(\widehat{\mathcal{B}}, \mu)$. By Observation 16.7 in [BBHU], the definable closure of $\text{fdcl}_{\mathbb{B}}(A)$ in $(\widehat{\mathcal{B}}, \mu)$ is equal to $\sigma(\text{fdcl}_{\mathbb{B}}(A))$, so $\text{dcl}_{\mathbb{B}}(A) = \sigma(\text{fdcl}_{\mathbb{B}}(A))$. Since $\text{fdcl}_{\mathbb{B}}(A)$ is a Boolean subalgebra of $\widehat{\mathcal{B}}$, $\text{cl}(\text{fdcl}_{\mathbb{B}}(A))$ is a Boolean subalgebra of $\widehat{\mathcal{B}}$. By metric completeness, $\text{cl}(\text{fdcl}_{\mathbb{B}}(A))$ is a σ -algebra and $\sigma(\text{fdcl}_{\mathbb{B}}(A))$ is closed, so $\text{cl}(\text{fdcl}_{\mathbb{B}}(A)) = \sigma(\text{fdcl}_{\mathbb{B}}(A))$. \square

Corollary 3.1.3. The only events that are definable without parameters in \mathcal{N} are \top and \perp .

Proof. For every first order sentence φ , either $T \models \varphi$ and $T^R \models \llbracket \varphi \rrbracket = \top$, or $T \models \neg\varphi$ and $T^R \models \llbracket \varphi \rrbracket = \perp$. So $\text{fdcl}_{\mathbb{B}}(\emptyset) = \{\top, \perp\}$. \square

3.2. First Order and Pointwise Definability. To prepare the way for a characterization of the definable elements of sort \mathbb{K} , we introduce two auxiliary notions, one that is stronger than definability in sort \mathbb{K} and one that is weaker than definability in sort \mathbb{K} . We will work in the nice randomization $\mathcal{P} = (\mathcal{K}, \mathcal{B})$ of \mathcal{M} , and let A be a subset of $\widehat{\mathcal{K}}$ and \mathbf{b} be an element of $\widehat{\mathcal{K}}$.

Definition 3.2.1. A first order formula $\varphi(u, \vec{v})$ is *functional* if

$$T \models (\forall \vec{v})(\exists^{\leq 1} u)\varphi(u, \vec{v}).$$

We say that \mathbf{b} is *first order definable on E over A* if there is a functional formula $\varphi(u, \vec{v})$ and a tuple $\vec{a} \in A^{<\mathbb{N}}$ such that $E = \llbracket \varphi(\mathbf{b}, \vec{a}) \rrbracket$.

We say that \mathbf{b} is *first order definable over A* , in symbols $\mathbf{b} \in \text{fdcl}(A)$, if \mathbf{b} is first order definable on \top over A .

Remarks 3.2.2. \mathbf{b} is first order definable over A if and only if there is a first order formula $\varphi(u, \vec{v})$ and a tuple \vec{a} from A such that

$$\mu(\llbracket (\forall u)(\varphi(u, \vec{a}) \leftrightarrow u = \mathbf{b}) \rrbracket) = 1.$$

First order definability has finite character, that is, \mathbf{b} is first order definable over A if and only if \mathbf{b} is first order definable over some finite subset of A .

If \mathbf{b} is first order definable on E over A , then E is first order definable over $A \cup \{\mathbf{b}\}$.

If \mathbf{b} is first order definable on \mathbf{D} over A , and \mathbf{E} is first order definable over $A \cup \{\mathbf{b}\}$, then \mathbf{b} is first order definable on $\mathbf{D} \sqcap \mathbf{E}$ over A .

Lemma 3.2.3. *If \mathbf{b} is first order definable over A then \mathbf{b} is definable over A in \mathcal{N} . Thus $\text{fdcl}(A) \subseteq \text{dcl}(A)$.*

Proof. Let $\mathcal{N}' \succ \mathcal{N}$ and suppose that $\text{tp}^{\mathcal{N}'}(\mathbf{b}) = \text{tp}^{\mathcal{N}'}(\mathbf{d})$. Then

$$\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket = \llbracket \varphi(\mathbf{d}, \vec{\mathbf{a}}) \rrbracket = \top.$$

Since φ is functional,

$$\llbracket (\forall t)(\forall u)(\varphi(t, \vec{\mathbf{a}}) \wedge \varphi(u, \vec{\mathbf{a}}) \rightarrow t = u) \rrbracket = \top.$$

Then $\llbracket \mathbf{b} = \mathbf{d} \rrbracket = \top$, so $\mathbf{b} = \mathbf{d}$, and by Fact 2.2.1, $\mathbf{b} \in \text{dcl}(A)$. \square

Definition 3.2.4. When A is countable, we define

$$\llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket := \{\omega \in \Omega : b(\omega) \in \text{dcl}^{\mathcal{M}}(A(\omega))\}.$$

Lemma 3.2.5. *If A is countable, then*

$$\llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket = \bigcup \{ \llbracket \theta(b, \vec{\mathbf{a}}) \rrbracket : \theta(u, \vec{v}) \text{ functional, } \vec{\mathbf{a}} \in A^{<\mathbb{N}} \},$$

and $\llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket \in \mathcal{B}$.

Proof. Note that for every first order formula $\theta(u, \vec{v})$, the formula

$$\theta(u, \vec{v}) \wedge (\exists^{\leq 1} u) \theta(u, \vec{v})$$

is functional. Therefore $\omega \in \llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket$ if and only if $b(\omega) \in \text{dcl}^{\mathcal{M}}(A(\omega))$, and this holds if and only if there is a functional formula $\theta(u, \vec{v})$ and a tuple $\vec{\mathbf{a}} \in A^{<\mathbb{N}}$ such that $\mathcal{M} \models \theta(b(\omega), \vec{\mathbf{a}}(\omega))$. Since A and L are countable, $\llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket$ is the union of countably many events in \mathcal{B} , and thus belongs to \mathcal{B} . \square

Definition 3.2.6. When A is countable, we say that \mathbf{b} is *pointwise definable over A* if

$$\mu(\llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket) = 1.$$

Corollary 3.2.7. *If A is countable, then \mathbf{b} is pointwise definable over A if and only if there is a function f on Ω such that:*

- (1) *For each $\omega \in \Omega$, $f(\omega)$ is a pair $\langle \theta_\omega(u, \vec{v}), \vec{\mathbf{a}}_\omega \rangle$ where $\theta_\omega(u, \vec{v})$ is functional and $\vec{\mathbf{a}}_\omega \in A^{|\vec{v}|}$;*
- (2) *f is $\sigma(\text{fdcl}_{\mathbb{B}}(A))$ -measurable (i.e., the inverse image of each point belongs to $\sigma(\text{fdcl}_{\mathbb{B}}(A))$);*
- (3) *$\mathcal{M} \models \theta_\omega(b(\omega), \vec{\mathbf{a}}_\omega(\omega))$ for almost every $\omega \in \Omega$.*

Proof. If $\omega \in \llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket$, let $f(\omega)$ be the first pair $\langle \theta_\omega, \vec{\mathbf{a}}_\omega \rangle$ such that $\theta_\omega(u, \vec{v})$ is functional, $\vec{\mathbf{a}}_\omega \in A^{|\vec{v}|}$, and $\mathcal{M} \models \theta_\omega(b(\omega), \vec{\mathbf{a}}_\omega(\omega))$. Otherwise let $f(\omega) = \langle \perp, \emptyset \rangle$. The result then follows from Lemma 3.2.5. \square

Lemma 3.2.8. *If \mathbf{b} is definable over A in \mathcal{N} , then \mathbf{b} is pointwise definable over some countable subset of A .*

Proof. By Fact 2.2.1 (3), we may assume that A is countable. By Lemma 3.2.5, the measure $r := \mu(\llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket)$ exists. Suppose \mathbf{b} is not pointwise definable over A . Then $r < 1$. For each finite collection $\chi_1(u, \vec{v}), \dots, \chi_n(u, \vec{v})$ of first order formulas, each tuple $\vec{\mathbf{a}} \in A^{<\mathbb{N}}$, and each $\omega \in \Omega \setminus \llbracket b \in \text{dcl}^{\mathcal{M}}(A) \rrbracket$, the sentence

$$(\exists u)[u \neq b(\omega) \wedge \bigwedge_{i=1}^n [\chi_i(b(\omega), \vec{\mathbf{a}}(\omega)) \leftrightarrow \chi_i(u, \vec{\mathbf{a}}(\omega))]$$

holds in \mathcal{M} , because $b(\omega)$ is not definable over $A(\omega)$. Therefore in \mathcal{P} we have

$$\mu(\llbracket (\exists u)[u \neq b \wedge \bigwedge_{i=1}^n [\chi_i(b, \vec{\mathbf{a}}) \leftrightarrow \chi_i(u, \vec{\mathbf{a}})] \rrbracket) \geq 1 - r.$$

By condition 2.1.1 (5), there is an element $\mathbf{d} \in \widehat{\mathcal{K}}$ such that

$$\mu(\llbracket \mathbf{d} \neq b \wedge \bigwedge_{i=1}^n [\chi_i(b, \vec{\mathbf{a}}) \leftrightarrow \chi_i(\mathbf{d}, \vec{\mathbf{a}})] \rrbracket) \geq 1 - r.$$

It follows that $\mu(\llbracket \mathbf{d} \neq b \rrbracket) \geq 1 - r$, and $\llbracket \chi_i(b, \vec{\mathbf{a}}) \rrbracket \doteq \llbracket \chi_i(\mathbf{d}, \vec{\mathbf{a}}) \rrbracket$ for each $i \leq n$. By compactness, in some elementary extension of \mathcal{N} there is an element \mathbf{d} such that $\mu(\llbracket \mathbf{d} \neq \mathbf{b} \rrbracket) \geq 1 - r$, and $\llbracket \chi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket = \llbracket \chi(\mathbf{d}, \vec{\mathbf{a}}) \rrbracket$ for each first order formula $\chi(u, \vec{v})$. Then $\mathbf{d} \neq \mathbf{b}$, and by quantifier elimination, $\text{tp}(\mathbf{d}/A) = \text{tp}(\mathbf{b}/A)$. Hence by Fact 2.2.1 (2), $\mathbf{b} \notin \text{dcl}(A)$. \square

The following example shows that the converse of Lemma 3.2.8 fails badly.

Example 3.2.9. Let \mathcal{M} be a finite structure with a constant symbol for every element. Then every element of \mathcal{K} is pointwise definable without parameters, but the only elements of $\widehat{\mathcal{K}}$ that are definable without parameters are the equivalence classes of constant functions $b: \Omega \rightarrow \mathcal{M}$.

3.3. Definability in Sort \mathbb{K} . We will now give necessary and sufficient conditions for an element of $\mathbf{b} \in \widehat{\mathcal{K}}$ to be definable over a parameter set $A \subseteq \widehat{K}$ in \mathcal{N} .

Theorem 3.3.1. \mathbf{b} is definable over A if and only if there exist pairwise disjoint events $\{\mathbf{E}_n: n \in \mathbb{N}\}$ such that $\sum_{n \in \mathbb{N}} \mu(\mathbf{E}_n) = 1$, and for each n , \mathbf{E}_n is definable over A , and \mathbf{b} is first order definable on \mathbf{E}_n over A .

Proof. (\Rightarrow): Suppose $\mathbf{b} \in \text{dcl}(A)$. By Lemma 3.2.8, \mathbf{b} is pointwise definable over some countable subset A_0 of A . The set of all events \mathbf{C} such that \mathbf{b} is first order definable on \mathbf{C} over A_0 is countable, and may be arranged in a list $\{\mathbf{C}_n: n \in \mathbb{N}\}$. Let $\mathbf{E}_0 = \mathbf{C}_0$, and

$$\mathbf{E}_{n+1} = \mathbf{C}_{n+1} \sqcap \neg(\mathbf{C}_0 \sqcup \dots \sqcup \mathbf{C}_n).$$

The events \mathbf{E}_n are pairwise disjoint, and for each n we have

$$\mathbf{E}_0 \sqcup \dots \sqcup \mathbf{E}_n = \mathbf{C}_0 \sqcup \dots \sqcup \mathbf{C}_n.$$

By Remarks 3.2.2, for each n , \mathbf{b} is first order definable on \mathbf{E}_n over A . By Lemma 3.2.5 and pointwise definability,

$$\sum_{n \in \mathbb{N}} \mu(\mathbf{E}_n) = \lim_{n \rightarrow \infty} \mu(\mathbf{C}_0 \sqcup \cdots \sqcup \mathbf{C}_n) = \mu(\llbracket \text{dcl}^{\mathcal{M}}(A_0) \rrbracket) = 1.$$

By Remarks 3.2.2, \mathbf{E}_n is definable over $A \cup \{\mathbf{b}\}$, and since \mathbf{b} is definable over A , \mathbf{E}_n is definable over A by Fact 2.2.3.

(\Leftarrow): Let \mathbf{E}_n be as in the theorem. For each n , we have $\mathbf{E}_n = \llbracket \theta_n(\mathbf{b}, \vec{\mathbf{a}}_n) \rrbracket$ for some functional formula θ_n and tuple $\vec{\mathbf{a}}_n \in A^{<\mathbb{N}}$. Since \mathbf{E}_n is definable over A , by Theorem 3.1.2 there is a sequence of formulas $\psi_k(\vec{v})$ and tuples $\vec{\mathbf{a}}_k \in A^{<\mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} d_{\mathbb{B}}(\llbracket \psi_k(\vec{\mathbf{a}}_k) \rrbracket, \llbracket \theta_n(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket) = 0.$$

Suppose \mathbf{d} has the same type over A as \mathbf{b} in some elementary extension \mathcal{N}' of \mathcal{N} . Then

$$\lim_{k \rightarrow \infty} d_{\mathbb{B}}(\llbracket \psi_k(\vec{\mathbf{a}}_k) \rrbracket, \llbracket \theta_n(\mathbf{d}, \vec{\mathbf{a}}) \rrbracket) = 0.$$

Hence

$$\llbracket \theta_n(\mathbf{d}, \vec{\mathbf{a}}_n) \rrbracket = \llbracket \theta_n(\mathbf{b}, \vec{\mathbf{a}}_n) \rrbracket = \mathbf{E}_n$$

in \mathcal{N}' . Since $\theta_n(u, \vec{v})$ is functional, we have $\llbracket \theta_n(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \subseteq \llbracket \mathbf{d} = \mathbf{b} \rrbracket$ for each n . Then

$$\mu(\llbracket \mathbf{d} = \mathbf{b} \rrbracket) \geq \sum_{n \in \mathbb{N}} \mu(\mathbf{E}_n) = 1,$$

so $\mathbf{d} = \mathbf{b}$. Then by Fact 2.2.1, $\mathbf{b} \in \text{dcl}(A)$. \square

Corollary 3.3.2. *An element $\mathbf{b} \in \widehat{\mathcal{K}}$ is definable without parameters if and only if \mathbf{b} is first order definable without parameters. Thus $\text{dcl}(\emptyset) = \text{fdcl}(\emptyset)$.*

Proof. (\Rightarrow): Suppose $\mathbf{b} \in \text{dcl}(\emptyset)$. By Theorem 3.3.1, there is an event \mathbf{E} such that $\mu(\mathbf{E}) > 0$, \mathbf{E} is definable without parameters, and \mathbf{b} is first order definable on \mathbf{E} without parameters. By Corollary 3.1.3 we have $\mathbf{E} = \top$, so \mathbf{b} is first order definable without parameters.

(\Leftarrow): By Lemma 3.2.3. \square

Corollary 3.3.3. *If $\text{fdcl}_{\mathbb{B}}(A)$ is finite, then $\text{dcl}_{\mathbb{B}}(A) = \text{fdcl}_{\mathbb{B}}(A)$ and $\text{dcl}(A) = \text{fdcl}(A)$.*

Proof. $\text{dcl}_{\mathbb{B}}(A) = \text{fdcl}_{\mathbb{B}}(A)$ follows from Theorem 3.1.2. Lemma 3.2.3 gives $\text{dcl}(A) \supseteq \text{fdcl}(A)$. For the other inclusion, suppose $\mathbf{b} \in \text{dcl}(A)$. By Theorem 3.3.1, there is a finite partition $\mathbf{E}_0, \dots, \mathbf{E}_k$ of \top , a tuple $\vec{\mathbf{a}} \in A^{<\mathbb{N}}$, and first order formulas $\psi_i(\vec{v})$ such that $\mathbf{E}_i = \llbracket \psi_i(\vec{\mathbf{a}}) \rrbracket$ and \mathbf{b} is first order definable on \mathbf{E}_i . Then there are functional formulas $\varphi_i(u, \vec{v})$ such that $\mathbf{E}_i \doteq \llbracket \varphi_i(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket$. We may take the formulas $\psi_i(\vec{v})$ to be pairwise inconsistent and such that

$T \models \bigvee_{i=0}^n \psi(\vec{v})$. Then $\bigwedge_{i=0}^n (\psi_i(\vec{v}) \rightarrow \varphi_i(u, \vec{v}))$ is a functional formula such that

$$\llbracket \bigwedge_{i=0}^n (\psi_i(\vec{a}) \rightarrow \varphi_i(\mathbf{b}, \vec{a})) \rrbracket = \top,$$

so \mathbf{b} is first order definable over A . \square

Corollary 3.3.4. *\mathbf{b} is definable over A if and only if:*

- (1) \mathbf{b} is pointwise definable over some countable subset of A ;
- (2) for each functional formula $\varphi(u, \vec{v})$ and tuple $\vec{a} \in A^{<\mathbb{N}}$, $\llbracket \varphi(\mathbf{b}, \vec{a}) \rrbracket$ is definable over A .

Proof. (\Rightarrow): Suppose $\mathbf{b} \in \text{dcl}(A)$. Then (1) holds by Lemma 3.2.8. $\llbracket \varphi(\mathbf{b}, \vec{a}) \rrbracket$ is obviously definable over $A \cup \{\mathbf{b}\}$, so $\llbracket \varphi(\mathbf{b}, \vec{a}) \rrbracket$ is definable over A by Fact 2.2.3, and thus (2) holds.

(\Leftarrow): Assume conditions (1) and (2). By (1) and Lemma 3.2.5, there is a sequence of functional formulas $\theta_n(u, \vec{v})$ and tuples $\vec{a}_n \in A^{<\mathbb{N}}$ such that

$$\llbracket \mathbf{b} \in \text{dcl}^{\mathcal{M}}(A) \rrbracket = \bigcup_{n \in \mathbb{N}} \llbracket \theta_n(\mathbf{b}, \vec{a}_n) \rrbracket \doteq \Omega.$$

Let $\mathbf{E}_n = \llbracket \theta_n(\mathbf{b}, \vec{a}_n) \rrbracket$, so \mathbf{b} is first order definable on \mathbf{E}_n over A . By Remark 3.2.2, we may take the \mathbf{E}_n to be pairwise disjoint, and thus $\sum_{n \in \mathbb{N}} \mu(\mathbf{E}_n) = 1$. By (2), \mathbf{E}_n is definable over A for each n . Then by Theorem 3.3.1, $\mathbf{b} \in \text{dcl}(A)$. \square

Corollary 3.3.5. *\mathbf{b} is definable over A if and only if:*

- (1) \mathbf{b} is pointwise definable over some countable subset of A ;
- (2) $\text{fdcl}_{\mathbb{B}}(A \cup \{\mathbf{b}\}) \subseteq \text{dcl}_{\mathbb{B}}(A)$.

Theorem 3.3.6. *\mathbf{b} is definable over A if and only if $\mathbf{b} = \lim_{m \rightarrow \infty} \mathbf{b}_m$, where each \mathbf{b}_m is first-order definable over A . Thus $\text{dcl}(A) = \text{cl}(\text{fdcl}(A))$.*

Proof. (\Rightarrow): Suppose that $\mathbf{b} \in \text{dcl}(A)$. If A is empty, then \mathbf{b} is already first order definable from A by Corollary 3.3.2. Assume A is not empty and let $\mathbf{c} \in A$. Let $\{\mathbf{E}_n : n \in \mathbb{N}\}$ be as in Theorem 3.3.1, and fix an $\varepsilon > 0$. Then for some n , $\sum_{k=0}^n \mu(\mathbf{E}_k) > 1 - \varepsilon$. For each k , \mathbf{E}_k is definable over A , so by Theorem 3.1.2, there is an event $\mathbf{D}_k \in \text{fdcl}_{\mathbb{B}}(A)$ such that $\mu(\mathbf{D}_k \triangle \mathbf{E}_k) < \varepsilon/n$. Since the events \mathbf{E}_k are pairwise disjoint, we may also take the events \mathbf{D}_k to be pairwise disjoint. We have $\mathbf{E}_k = \llbracket \theta_k(\mathbf{b}, \vec{a}_k) \rrbracket$ for some functional $\theta_k(u, \vec{v})$, so we may assume that \mathbf{D}_k has the additional properties that $\mathbf{D}_k \sqsubseteq \llbracket (\exists! u) \theta_k(u, \vec{a}_k) \rrbracket$, and that $\mathbf{D}_k = \llbracket \psi_k(\vec{a}_k) \rrbracket$ for some formula $\psi_k(\vec{v})$. Then there is a unique element $\mathbf{d} \in \widehat{\mathcal{K}}$ such that

$$\begin{cases} \mathcal{M} \models \theta_k(\mathbf{d}(\omega), \vec{a}_k(\omega)) & \text{if } k \leq n \text{ and } \omega \in \llbracket \psi_k(\vec{a}_k) \rrbracket, \\ \mathbf{d}(\omega) = \mathbf{c}(\omega) & \text{if } \omega \in \Omega \setminus \bigcup_{k=0}^n \llbracket \psi_k(\vec{a}_k) \rrbracket. \end{cases}$$

Then \mathbf{d} is first order definable over A , and $d_{\mathbb{K}}(\mathbf{b}, \mathbf{d}) < \varepsilon$.

(\Leftarrow): This follows because first order definability implies definability (Lemma 3.2.3) and the set $\text{dcl}(A)$ is metrically closed (Fact 2.2.3 (2)). \square

The following result was proved in [Be] by an indirect argument using Lascar types. We give a simple direct proof here.

Proposition 3.3.7. *For any model $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$ of T^R and set $A \subseteq \widehat{\mathcal{K}}$, $\text{acl}_{\mathbb{B}}(A) = \text{dcl}_{\mathbb{B}}(A)$ and $\text{acl}(A) = \text{dcl}(A)$.*

Proof. By Facts 2.2.1 and 2.2.2, we may assume \mathcal{N} is \aleph_1 -saturated and A is countable. Suppose an event $E \in \widehat{\mathcal{B}}$ is not definable over A . By Fact 2.2.1 and \aleph_1 -saturation there exists $D \in \widehat{\mathcal{B}}$ such that $\text{tp}(D/A) = \text{tp}(E/A)$ but $d_{\mathbb{B}}(D, E) > 0$. By \aleph_1 -saturation again, there is a countable sequence of events $\langle F_n : n \in \mathbb{N} \rangle$ in $\widehat{\mathcal{B}}$ such that

$$\mu(C \cap F_n) = \mu(C \setminus F_n) = \mu(C)/2$$

for each n and each event C in the Boolean algebra generated by

$$\text{fdcl}_{\mathbb{B}}(A) \cup \{D, E\} \cup \{F_k : k < n\}.$$

For each n , let

$$D_n = (D \cap F_n) \cup (E \setminus F_n).$$

Then for each $C \in \text{fdcl}_{\mathbb{B}}(A)$ and $n \in \mathbb{N}$, we have

$$\mu(D_n \cap C) = \mu(D \cap C)/2 + \mu(E \cap C)/2 = \mu(E \cap C).$$

By quantifier elimination, $\text{tp}(D_n/A) = \text{tp}(E/A)$ for each $n \in \mathbb{N}$. Moreover, whenever $k < n$ we have

$$D_n \setminus D_k = ((D \setminus D_k) \cap F_n) \cup ((E \setminus D_k) \setminus F_n),$$

so

$$\mu(D_n \setminus D_k) = \mu(D \setminus D_k)/2 + \mu(E \setminus D_k)/2.$$

Note that whenever $\text{tp}(D'/A) = \text{tp}(D''/A)$, we have $\mu(D') = \mu(D'')$, and hence

$$\mu(D' \setminus D'') = \mu(D'' \setminus D') = d_{\mathbb{B}}(D', D'')/2.$$

Therefore

$$d_{\mathbb{B}}(D_n, D_k) = d_{\mathbb{B}}(D, D_k)/2 + d_{\mathbb{B}}(E, D_k)/2 \geq d_{\mathbb{B}}(D, E)/2.$$

It follows that the set of realizations of $\text{tp}(E/A)$ is not compact, and E is not algebraic over A . This shows that $\text{acl}_{\mathbb{B}}(A) = \text{dcl}_{\mathbb{B}}(A)$.

Now suppose $\mathbf{b} \in \text{acl}(A) \setminus \text{dcl}(A)$. There is an element $\mathbf{c} \in \widehat{\mathcal{K}}$ such that $\text{tp}(\mathbf{b}/A) = \text{tp}(\mathbf{c}/A)$ but $d_{\mathbb{K}}(\mathbf{b}, \mathbf{c}) > 0$. For each first order formula $\psi(u, \vec{v})$ and $\vec{\mathbf{a}} \in A^{<\mathbb{N}}$, $\llbracket \psi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \in \text{acl}_{\mathbb{B}}(\{\mathbf{b}\} \cup A) \subseteq \text{acl}_{\mathbb{B}}(\text{acl}(A))$. By Fact 2.2.3, $\llbracket \psi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \in \text{acl}_{\mathbb{B}}(A)$. By the preceding paragraph, $\llbracket \psi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \in \text{dcl}_{\mathbb{B}}(A)$. Since $\text{tp}(\mathbf{b}/A) = \text{tp}(\mathbf{c}/A)$, we have $\text{tp}(\llbracket \psi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket/A) = \text{tp}(\llbracket \psi(\mathbf{c}, \vec{\mathbf{a}}) \rrbracket/A)$. By Fact 2.2.1, it follows that $\llbracket \psi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket = \llbracket \psi(\mathbf{c}, \vec{\mathbf{a}}) \rrbracket$ for every first order formula $\psi(u, \vec{v})$. Then $\text{tp}(b(\omega)/A(\omega)) = \text{tp}(c(\omega)/A(\omega))$ for μ -almost all ω . By \aleph_1 -saturation, there are countably many independent events $D_n \in \widehat{\mathcal{B}}$ such that $D_n \subseteq \llbracket \mathbf{b} \neq \mathbf{c} \rrbracket$ and $\mu(D_n) = d_{\mathbb{K}}(\mathbf{b}, \mathbf{c})/2$. Let \mathbf{c}_n agree with \mathbf{c} on D_n and agree with \mathbf{b} elsewhere. We have $\text{tp}(\mathbf{c}_n/A) = \text{tp}(\mathbf{b}/A)$ for every $n \in \mathbb{N}$, and

$d_{\mathbb{K}}(\mathbf{c}_n, \mathbf{c}_k) = d_{\mathbb{K}}(\mathbf{b}, \mathbf{c})/2$ whenever $k < n$. Thus the set of realizations of $\text{tp}(\mathbf{b}/A)$ is not compact, contradicting the fact that $\mathbf{b} \in \text{acl}(A)$. \square

4. A SPECIAL CASE: \aleph_0 -CATEGORICAL THEORIES

4.1. Definability and \aleph_0 -Categoricity. We use our preceding results to characterize \aleph_0 -categorical theories in terms of definability in randomizations.

Theorem 4.1.1. *The following are equivalent:*

- (1) T is \aleph_0 -categorical;
- (2) $\text{fdcl}_{\mathbb{B}}(A)$ is finite for every finite A ;
- (3) $\text{dcl}_{\mathbb{B}}(A)$ is finite for every finite A ;
- (4) $\text{fdcl}_{\mathbb{B}}(A) = \text{dcl}_{\mathbb{B}}(A)$ for every finite A ;
- (5) $\text{fdcl}(A)$ is finite for every finite A ;
- (6) $\text{dcl}(A)$ is finite for every finite A ;
- (7) $\text{fdcl}(A) = \text{dcl}(A)$ for every finite A ;

Proof. By the Ryll-Nardzewski Theorem (see [CK], Theorem 2.3.13), (1) is equivalent to

(0) For each n there are only finitely many formulas in n variables up to T -equivalence.

Assume (0) and let $A \subseteq \widehat{\mathcal{K}}$ be finite. Then (2) holds. Moreover, there are only finitely many functional formulas in $|A| + 1$ variables, so (5) holds. Then by Corollary 3.3.3, (3), (4), (6), and (7) hold.

Now assume that (0) fails.

Proof that (2) and (3) fail: For some n there are infinitely many formulas in n variables that are not T -equivalent. Hence there is an n -type p in T without parameters that is not isolated. So there are formulas $\varphi_1(\vec{v}), \varphi_2(\vec{v}), \dots$ in p such that for each $k > 0$, $T \models \varphi_{k+1} \rightarrow \varphi_k$ but the formula $\theta_k = \varphi_k \wedge \neg \varphi_{k+1}$ is consistent with T . The formulas θ_k are consistent but pairwise inconsistent. By Fullness, for each $k > 0$ there exists an n -tuple $\vec{\mathbf{b}}_k \in \widehat{\mathcal{K}}^n$ such that $\llbracket \theta_k(\vec{\mathbf{b}}_k) \rrbracket = \top$. Since the measured algebra $(\widehat{\mathcal{B}}, \mu)$ is atomless, there are pairwise disjoint events $\mathbf{E}_1, \mathbf{E}_2, \dots$ in $\widehat{\mathcal{B}}$ such that $\mu(\mathbf{E}_k) = 2^{-k}$ for each $k > 0$. By applying Lemma 2.1.8 k times, we see that for each $k > 0$ there is an n -tuple $\vec{\mathbf{a}}_k \in \widehat{\mathcal{K}}^n$ that agrees with $\vec{\mathbf{b}}_i$ on \mathbf{E}_i whenever $0 < i \leq k$. Whenever $0 < k \leq j$, we have $\mu(\llbracket \vec{\mathbf{a}}_k = \vec{\mathbf{a}}_j \rrbracket) \geq 1 - 2^{-k}$. So $\langle \vec{\mathbf{a}}_1, \vec{\mathbf{a}}_2, \dots \rangle$ is a Cauchy sequence, and by metric completeness the limit $\vec{\mathbf{a}} = \lim_{k \rightarrow \infty} \vec{\mathbf{a}}_k$ exists in $\widehat{\mathcal{K}}^n$. Let $A = \text{range}(\vec{\mathbf{a}})$. For each $k > 0$ we have $\mathbf{E}_k = \llbracket \vec{\mathbf{a}} = \vec{\mathbf{b}}_k \rrbracket = \llbracket \theta_k(\vec{\mathbf{a}}) \rrbracket$, so $\mathbf{E}_k \in \text{fdcl}_{\mathbb{B}}(A)$. Thus $\text{fdcl}_{\mathbb{B}}(A)$ is infinite, so (2) fails and (3) fails.

Proof that (4) fails: Let \mathbf{E}_k be as in the preceding paragraph. The set $\text{fdcl}_{\mathbb{B}}(A)$ is countable. But the closure $\text{cl}(\text{fdcl}_{\mathbb{B}}(A))$ is uncountable, because for each set $S \subseteq \mathbb{N} \setminus \{0\}$, the supremum $\bigsqcup_{k \in S} \mathbf{E}_k$ belongs to $\text{cl}(\text{fdcl}_{\mathbb{B}}(A))$. Thus by Theorem 3.1.2,

$$\text{dcl}_{\mathbb{B}}(A) = \text{cl}(\text{fdcl}_{\mathbb{B}}(A)) \neq \text{fdcl}_{\mathbb{B}}(A),$$

and (4) fails.

Proof that (5), (6), and (7) fail: By Corollary 2.1.6, there exist $\mathbf{c}, \mathbf{d} \in \mathcal{K}$ such that $\llbracket \mathbf{c} \neq \mathbf{d} \rrbracket = \top$. Let C be the finite set $C = A \cup \{\mathbf{c}, \mathbf{d}\}$. By Remark 2.2.4, for any event $D \in \text{fdcl}_{\mathbb{B}}(A)$, the characteristic function 1_D of D with respect to \mathbf{c}, \mathbf{d} is definable over C . Moreover, we always have $d_{\mathbb{K}}(1_D, 1_E) = d_{\mathbb{B}}(D, E)$. It follows that $\text{fdcl}(C)$ is infinite, so (5) and (6) fail. To show that (7) fails, we take an event $D \in \text{dcl}_{\mathbb{B}}(A) \setminus \text{fdcl}_{\mathbb{B}}(A)$. By Theorem 3.1.2 we have $D \in \text{cl}(\text{fdcl}_{\mathbb{B}}(A))$. It follows that $1_D \in \text{cl}(\text{fdcl}(C))$, so by Theorem 3.3.6, $1_D \in \text{dcl}(C)$. Hence $\text{dcl}(C)$ is uncountable. But $\text{fdcl}(C)$ is countable, so (7) fails. \square

By the Ryll-Nardzewski Theorem, if T is \aleph_0 -categorical then for each n , T has finitely many n -types; so each type p in the variables (u, \vec{v}) has an *isolating formula*, that is, a formula $\varphi(u, \vec{v})$ such that $T \models \varphi(u, \vec{v}) \leftrightarrow \bigwedge p$.

We now characterize the definable closure of a finite set $A \subseteq \widehat{\mathcal{K}}$ in the case that T is \aleph_0 -categorical. Hereafter, when A is a finite subset of $\widehat{\mathcal{K}}$, $\vec{\mathbf{a}}$ will denote a finite tuple whose range is A .

Corollary 4.1.2. *Suppose that T is \aleph_0 -categorical, $\mathbf{b} \in \widehat{\mathcal{K}}$, and A is a finite subset of $\widehat{\mathcal{K}}$. Then $\mathbf{b} \in \text{dcl}(A)$ if and only if:*

- (1) \mathbf{b} is pointwise definable over A ;
- (2) for every isolating formula $\varphi(u, \vec{v})$, if $\mu(\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket) > 0$ then

$$\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket = \llbracket (\exists u)\varphi(u, \vec{\mathbf{a}}) \rrbracket.$$

Proof. (\Rightarrow): Suppose $\mathbf{b} \in \text{dcl}(A)$. (1) holds by Lemma 3.2.8. Suppose $\varphi(u, \vec{v})$ is isolating and $\mu(\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket) > 0$. We have $\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \in \text{fdcl}_{\mathbb{B}}(\{\mathbf{b}\} \cup A)$, so by Corollary 3.3.5, $\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \in \text{dcl}_{\mathbb{B}}(A)$. By Theorem 4.1.1, $\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \in \text{fdcl}_{\mathbb{B}}(A)$. We note that $(\exists u)\varphi(u, \vec{v})$ is an isolating formula, so $\llbracket (\exists u)\varphi(u, \vec{\mathbf{a}}) \rrbracket$ is an atom of $\text{fdcl}_{\mathbb{B}}(A)$. Therefore (2) holds.

(\Leftarrow): Assume (1) and (2). By (2), for every isolating formula $\varphi(u, \vec{v})$ such that $\mu(\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket) > 0$, we have

$$\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket \in \text{fdcl}_{\mathbb{B}}(A).$$

Every formula $\theta(u, \vec{v})$ is T -equivalent to a finite disjunction of isolating formulas in the variables (u, \vec{v}) . It follows that $\text{fdcl}_{\mathbb{B}}(A \cup \{\mathbf{b}\}) \subseteq \text{fdcl}_{\mathbb{B}}(A)$. Therefore by Corollary 3.3.5, $\mathbf{b} \in \text{dcl}(A)$. \square

Corollary 4.1.3. *Suppose that T is \aleph_0 -categorical, $\mathbf{b} \in \widehat{\mathcal{K}}$, and A is a finite subset of $\widehat{\mathcal{K}}$. Then $\mathbf{b} \in \text{dcl}(A)$ if and only if for every isolating formula $\psi(\vec{v})$ there is a functional formula $\varphi(u, \vec{v})$ such that $\llbracket \psi(\vec{\mathbf{a}}) \rrbracket \subseteq \llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket$.*

Proof. (\Rightarrow): Suppose $\mathbf{b} \in \text{dcl}(A)$. By Theorem 4.1.1, \mathbf{b} is first order definable over $\vec{\mathbf{a}}$, so there is a functional formula $\varphi(u, \vec{v})$ such that $\llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket = \top$. Then for every isolating $\psi(\vec{v})$ we have $\llbracket \psi(\vec{\mathbf{a}}) \rrbracket \subseteq \llbracket \varphi(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket$.

(\Leftarrow): There is a finite set $\{\psi_0(\vec{v}), \dots, \psi_k(\vec{v})\}$ that contains exactly one isolating formula for each $|\vec{\mathbf{a}}|$ -type of T . By hypothesis, for each $i \leq k$ there is a functional formula $\varphi_i(u, \vec{v})$ such that $\llbracket \psi_i(\vec{\mathbf{a}}) \rrbracket \subseteq \llbracket \varphi_i(\mathbf{b}, \vec{\mathbf{a}}) \rrbracket$. Since the

formulas $\psi_i(\vec{v})$ are pairwise inconsistent, the formula $\bigvee_{i=0}^k (\psi_i(\vec{v}) \wedge \varphi_i(u, \vec{v}))$ is functional, and

$$\llbracket \bigvee_{i=0}^k (\psi_i(\vec{a}) \wedge \varphi_i(\mathbf{b}, \vec{a})) \rrbracket = \top.$$

Hence \mathbf{b} is first order definable over \vec{a} , so by Lemma 3.2.3 we have $\mathbf{b} \in \text{dcl}(A)$. \square

4.2. The Theory DLO^R . We will use Corollary 4.1.3 to give a more natural characterization of the definable closure of a finite parameter set in a model of DLO^R , where DLO is the theory of dense linear order without endpoints. Note that in DLO, every type in (v_1, \dots, v_n) has an isolating formula of the form $\bigwedge_{i=1}^{n-1} u_i \alpha_i u_{i+1}$ where $\{u_1, \dots, u_n\} = \{v_1, \dots, v_n\}$ and each $\alpha_i \in \{<, =\}$. (This formula linearly orders the equality-equivalence classes).

Corollary 4.2.1. *Let $T = \text{DLO}$, $\mathbf{b} \in \widehat{\mathcal{K}}$, and A be a finite subset of $\widehat{\mathcal{K}}$. Then $\mathbf{b} \in \text{dcl}(A)$ if and only if for every isolating formula $\psi(v_1, \dots, v_n)$ there is an $i \in \{1, \dots, n\}$ such that $\llbracket \psi(\vec{a}) \rrbracket \sqsubseteq \llbracket \mathbf{b} = \mathbf{a}_i \rrbracket$.*

Proof. For any $\mathcal{M} \models \text{DLO}$ and parameter set A , we have $\text{dcl}^{\mathcal{M}}(A) = A$. Therefore for every isolating formula $\psi(v_1, \dots, v_n)$ and functional formula $\varphi(u, v_1, \dots, v_n)$ there exists $i \in \{1, \dots, n\}$ such that

$$\text{DLO} \models (\psi(v_1, \dots, v_n) \wedge \varphi(u, v_1, \dots, v_n)) \rightarrow u = v_i.$$

The result now follows from Corollary 4.1.3. \square

In the theory DLO, we define $\min(u, v)$ and $\max(u, v)$ in the usual way. For $\mathbf{a}, \mathbf{b} \in \widehat{\mathcal{K}}$, we let $\min(\mathbf{a}, \mathbf{b})$ be the unique element $\mathbf{e} \in \widehat{\mathcal{K}}$ such that

$$\llbracket \mathbf{e} = \min(\mathbf{a}, \mathbf{b}) \rrbracket = \top,$$

and similarly for \max . For finite subsets A of $\widehat{\mathcal{K}}$, $\min(A)$ and $\max(A)$ are defined by repeating the two-variable functions \min and \max in the natural way.

We next show that in DLO^R , the definable closure of a finite set can be characterized as the closure under a “choosing function” of four variables.

Definition 4.2.2. In the theory DLO, let ℓ be the function of four variables defined by the condition

$$\ell(u, v, x, y) = x \text{ if } u < v, \text{ and } \ell(u, v, x, y) = y \text{ if not } u < v.$$

For $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathcal{K}$, let $\ell(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ be the unique element $\mathbf{e} \in \widehat{\mathcal{K}}$ such that $\llbracket \mathbf{e} = \ell(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \rrbracket = \top$. Given a set $A \subseteq \widehat{\mathcal{K}}$, let $\text{lcl}(A)$ be the closure of A under the function ℓ .

Note that in DLO, the function ℓ is definable without parameters. In both DLO and DLO^R , $\min(u, v) = \ell(u, v, u, v)$, and $\max(u, v) = \ell(u, v, v, u)$.

Proposition 4.2.3. *Let $T = \text{DLO}$. Then for every finite subset A of $\widehat{\mathcal{K}}$, $\text{dcl}(A) = \text{lcl}(A)$.*

Proof. It is clear that $\text{lcl}(A) \subseteq \text{dcl}(A)$.

We prove the other inclusion. If A is empty, the result is trivial, so we assume A is non-empty. Let $\mathbf{0} = \min(A)$, $\mathbf{1} = \max(A)$. We have $\mathbf{0}, \mathbf{1} \in \text{lcl}(A)$. Let $\Omega_0 = \llbracket 0 < 1 \rrbracket$. Note that $\Omega \setminus \Omega_0 = \llbracket 0 = 1 \rrbracket$. If $\mu(\Omega_0) = 0$, then A is a singleton, and we trivially have $\text{lcl}(A) = \text{dcl}(A) = A$. We may therefore assume that $\mu(\Omega_0) > 0$. To simplify notation we will instead assume that $\Omega_0 = \Omega$; the argument in the general case is similar.

In the following, all characteristic functions are understood to be with respect to $\mathbf{0}, \mathbf{1}$. Note that $\ell(\mathbf{a}, \mathbf{b}, \mathbf{0}, \mathbf{1})$ is the characteristic function of the event $\llbracket \mathbf{a} < \mathbf{b} \rrbracket$. If \mathbf{d} is the characteristic function of an event D and \mathbf{e} is the characteristic function of an event E , then $\ell(\mathbf{d}, \mathbf{1}, \mathbf{1}, \mathbf{0})$ is the characteristic function of $\neg D$, $\min(\mathbf{d}, \mathbf{e})$ is the characteristic function of $D \cap E$, and $\max(\mathbf{d}, \mathbf{e})$ is the characteristic function of $D \sqcup E$. It follows that for every quantifier-free first order formula $\varphi(\vec{v})$ of DLO with $|\vec{v}| = |\vec{\mathbf{a}}|$, the characteristic function of the event $\llbracket \varphi(\vec{\mathbf{a}}) \rrbracket$ belongs to $\text{lcl}(A)$. Since DLO admits quantifier elimination, the characteristic function of every event that is first order definable over A belongs to $\text{lcl}(A)$. Hence by Theorem 4.1.1, the characteristic function of every event in $\text{dcl}_{\mathbb{B}}(A)$ belongs to $\text{lcl}(A)$. Moreover, for every $\mathbf{c} \in A$ and event $D \in \text{dcl}_{\mathbb{B}}(A)$ with characteristic function \mathbf{d} , $\mathbf{c} \upharpoonright D := \ell(\mathbf{d}, \mathbf{1}, \mathbf{0}, \mathbf{c})$ is the element that agrees with \mathbf{c} on D and agrees with $\mathbf{0}$ on the complement of D , so $\mathbf{c} \upharpoonright D$ belongs to $\text{lcl}(A)$. Let $\{D_1, \dots, D_n\}$ be the set of atoms of $\text{dcl}_{\mathbb{B}}(A)$ (which is finite because DLO is \aleph_0 -categorical). By Corollary 4.2.1, every element of $\text{dcl}(A)$ has the form

$$\max(\mathbf{c}_1 \upharpoonright D_1, \dots, \mathbf{c}_n \upharpoonright D_n)$$

for some $\mathbf{c}_1, \dots, \mathbf{c}_n \in A$. Therefore $\text{dcl}(A) \subseteq \text{lcl}(A)$. \square

Example 4.2.4. In this example we show that the exchange property fails for DLO^R , even though it holds for DLO. Thus the exchange property is not preserved under randomizations. Let $T = \text{DLO}$. By Fullness, there exist elements $\mathbf{a}, \mathbf{b} \in \widehat{\mathcal{K}}$ such that $\max(\mathbf{a}, \mathbf{b}) \notin \{\mathbf{a}, \mathbf{b}\}$. Let $\mathbf{c} = \max(\mathbf{a}, \mathbf{b})$, $\mathbf{d} = \min(\mathbf{a}, \mathbf{b})$. It is easy to check that

$$\text{dcl}(\{\mathbf{a}, \mathbf{b}\}) = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}, \quad \text{dcl}(\{\mathbf{a}, \mathbf{c}\}) = \{\mathbf{a}, \mathbf{c}\}, \quad \text{dcl}(\{\mathbf{a}\}) = \{\mathbf{a}\}.$$

Thus $\mathbf{c} \in \text{dcl}(\{\mathbf{a}, \mathbf{b}\}) \setminus \text{dcl}(\{\mathbf{a}\})$ but $\mathbf{b} \notin \text{dcl}(\{\mathbf{a}, \mathbf{c}\})$.

REFERENCES

- [AK] Uri Andrews and H. Jerome Keisler. Randomizations of Theories with Countably Many Countable Models. To appear. Available online at www.math.wisc.edu/~Keisler.
- [Be] Itai Ben Yaacov. On Theories of Random Variables. To appear, Israel J. Math. arXiv:0901.1584v3 (2001).
- [BBHU] Itai Ben Yaacov, Alexander Berenstein, C. Ward Henson and Alexander Usvyatsov. Model Theory for Metric Structures. To appear, Lecture Notes of the London Math. Society.

[BK] Itai Ben Yaacov and H. Jerome Keisler. Randomizations of Models as Metric Structures. *Confluentes Mathematici* 1 (2009), pp. 197-223.

[BU] Itai Ben Yaacov and Alexander Usvyatsov. Continuous first order logic and local stability. *Transactions of the American Mathematical Society* 362 (2010), no. 10, 5213-5259.

[CK] C.C.Chang and H. Jerome Keisler. *Model Theory*. Dover 2012.

[EG] Clifton Early and Isaac Goldbring. Thorn-Forking in Continuous Logic. *Journal of Symbolic Logic* 77 (2012), 63-93.

[Go1] Isaac Goldbring. Definable Functions in Urysohn's Metric Space. To appear, *Illinois Journal of Mathematics*.

[Go2] Isaac Goldbring. An Approximate Herbrand's Theorem and Definable Functions in Metric Structures. *Math. Logic Quarterly* 50 (2012), 208-216.

[Go3] Isaac Goldbring. Definable Operators on Hilbert Spaces. *Notre Dame Journal of Formal Logic* 53 (2012), 193-201.

[GL] Isaac Goldbring and Vinicius Lopes. Pseudofinite and Pseudocompact Metric Structures. To appear, *Notre Dame Journal of Formal Logic*. Available online at www.homepages.math.uic.edu/~isaac.

[Ke1] H. Jerome Keisler. Randomizing a Model. *Advances in Math* 143 (1999), 124-158.

[Ke2] H. Jerome Keisler. Separable Randomizations of Models. To appear. Available online at www.math.wisc.edu/~Keisler.

UNIVERSITY OF WISCONSIN-MADISON, DEPARTMENT OF MATHEMATICS, MADISON, WI 53706-1388

E-mail address: andrews@math.wisc.edu

URL: www.math.wisc.edu/~andrews

E-mail address: keisler@math.wisc.edu

URL: www.math.wisc.edu/~keisler

UNIVERSITY OF ILLINOIS AT CHICAGO, DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, SCIENCE AND ENGINEERING OFFICES (M/C 249), 851 S. MORGAN ST., CHICAGO, IL 60607-7045, USA

E-mail address: isaac@math.uic.edu

URL: www.math.uic.edu/~isaac