

A New Spectrum of Recursive Models Using An Amalgamation Construction

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Abstract

We employ an infinite-signature Hrushovski amalgamation construction to yield two results in Recursive Model Theory. The first result, that there exists a strongly minimal theory whose only recursively presentable models are the prime and saturated models, adds a new *spectrum* to the list of known possible spectra. The second result, that there exists a strongly minimal theory in a finite language whose only recursively presentable model is saturated, gives the second non-trivial example of a spectrum produced in a finite language.

1 Introduction

Baldwin and Lachlan [2] developed the theory of \aleph_1 -categoricity in terms of strongly minimal sets. They show, in particular, that for any \aleph_1 -categorical theory in a countable language which is not \aleph_0 -categorical, the countable models form a chain of elementary embeddings of length $\omega+1$: $\mathcal{M}_0 \prec \mathcal{M}_1 \prec \dots \prec \mathcal{M}_\omega$, where \mathcal{M}_0 is the prime model and \mathcal{M}_ω is the saturated model. Furthermore, there is a strongly minimal formula such that each \mathcal{M}_i is characterized by the size of a maximal algebraically independent subset realizing that formula. In particular, the models of a strongly minimal theory are characterized by the size of a maximal algebraically independent subset. We call such a subset a *basis* for the model.

Let T be an \aleph_1 -categorical theory in a countable language which is not \aleph_0 -categorical. We have, by Baldwin-Lachlan, the chain of countable models of T : $\mathcal{M}_0 \prec \mathcal{M}_1 \prec \dots \prec \mathcal{M}_\omega$. We associate to T the *Spectrum of Recursive Models of T* , $SRM(T) = \{i | \mathcal{M}_i \text{ has a recursive presentation}\}$. We call a set $S \subseteq \omega + 1$ a *spectrum* if there exists an \aleph_1 -categorical T such that $SRM(T) = S$.

A general problem of recent years has been to characterize which subsets of $\omega + 1$ are spectra. Many results in this direction have been of the form “ S is a spectrum” and have been proved by providing a construction of a theory T yielding S as $SRM(T)$. For example, Goncharov [3] showed that $\{0\}$ is a spectrum, and Kudaibergenov [8] generalized Goncharov’s method to show that $\{0, \dots, n\}$ is a spectrum for any n . In [5], it is shown that $\{\omega\}$ is a spectrum, and in [7], it is shown that ω and $\omega + 1 \setminus \{0\}$ are both spectra, and Nies [9] shows that $\{1, \dots, n\}$ is a spectrum for any natural number n .

One thing that these examples all have in common is that each spectrum is an interval within $\omega + 1$, leading some to make the conjecture that this is always the case. In Theorem 1, we provide a counterexample to this conjecture.

Theorem 1. *There exists a strongly minimal theory T such that $SRM(T) = \{0, \omega\}$.*

There is also the question as to which of the possible spectra can be achieved with a finite language. That is, for which $S \subseteq \omega + 1$ does there exist a theory T in a finite language such that $SRM(T) = S$? This question has its roots in the paper of Herwig, Lempp, and Ziegler [4], where they show that $\{0\}$ can be achieved as a spectrum of an \aleph_1 -categorical theory in a finite language. In [1], this result is extended to show that $\{0, \dots, n\}$ can be achieved as a spectrum of an \aleph_1 -categorical theory in a finite language for any n , and similarly for the set ω . In Theorem 2, we present another such result.

Theorem 2. *There exists a strongly minimal theory T in a finite language such that $SRM(T) = \{\omega\}$*

In [1], as in this paper, the construction proceeds via an alteration of the Hrushovski construction. Aside from this similarity, the content of the proofs are different. In [1], the proof proceeds via encoding information into the type of a set based on how independent the set is, yielding that a truly independent $n + 1$ -tuple codes non-recursive information. In the proof of Theorem 2, we do not directly code information, but rather we infinitely often change our mind about which formulas are algebraic to entice a recursive model to generate an independent sequence of elements.

Both main theorems will proceed via an alteration of the Hrushovski construction (see [6]) to allow for infinite languages. Through section 2 and most of section 3, we will work over a general countable language and allow for any μ function, to demonstrate the construction in generality. The amalgamation method, as developed by Hrushovski in [6], will be followed closely to ensure that the theory resulting from the construction is in fact strongly minimal. The new content in sections 2 and 3 is confined to the use of an infinite language and to the use of recursion theoretic information to change algebraicity within the Hrushovski construction. Nonetheless, we state the core lemmas and some chosen proofs in the interest of self-containment of this paper.

2 The Amalgamation Class

Let L be a countable relational language. Though the amalgamation construction described here works with any relational language, we work with a language whose symbols are all ternary. Throughout the construction, we enforce that each relation is symmetric and holds only on distinct triples. For a finite L -structure A and relation symbol $R \in L$, we write $|R(A)|$ for the number of triples from the set A on which R holds, counting each triple only once.

Definition 3. Set $\delta : \{\text{finite } L\text{-structures}\} \rightarrow \mathbb{Z} \cup \{-\infty\}$ by $\delta(A) = |A| - \sum_{R \in L} |R(A)|$

From δ , we make the standard definitions for a Hrushovski construction:

Definition 4. For any finite L -structures A and B and infinite L -structure D , we define:

- $\delta(B/A) = \delta(A \cup B) - \delta(A)$.
- If $A \subseteq B$, we set $\delta(A, B) = \min\{\delta(C) \mid A \subseteq C \subseteq B\}$.
- If $A \subseteq B$, we say A is strong in B or $A \leq B$ if $\delta(A) = \delta(A, B)$.
We say A is strong in D if $A \subseteq D$ and A is strong in C for each finite $A \subseteq C \subseteq D$.
- We say B is simply algebraic over A if $B \neq \emptyset$, $A \cap B = \emptyset$, $A \leq A \cup B$, $\delta(B/A) = 0$, and there is no proper subset B' of B such that $\delta(B'/A) = 0$.

- We say that B is *minimally simply algebraic* over A if B is simply algebraic over A and there is no proper subset A' of A such that B is simply algebraic over A' .

The standard necessary property of pre-dimension functions holds here, namely that $\delta(A \cup B) \leq \delta(A) + \delta(B) - \delta(A \cap B)$. We call this property *sub-modularity*. This is verified by seeing that each relation is counted at least as many times on the left as on the right. Also, note that equality holds if and only if there are no relations holding on tuples from $(A \cup B)$ other than those already holding in A or B . In this case, we say A and B are *freely joined* over $A \cap B$. From the sub-modularity property, the following standard lemmas hold.

Lemma 5. *Suppose $A \leq N$. Then:*

1. $\delta(X \cap A) \leq \delta(X)$ for any $X \subseteq N$
2. $\delta(A', A) = \delta(A', N)$ for any $A' \subseteq A$
3. $A' \leq A \leq N$ implies $A' \leq N$

Proof. 2 and 3 follow from 1 trivially, so we will prove 1. $\delta(X \cup A) \leq \delta(X) + \delta(A) - \delta(X \cap A)$. Reordering, we get $\delta(X \cap A) \leq \delta(X) + \delta(A) - \delta(X \cup A) \leq \delta(X)$, where the last inequality holds because $A \leq N$. \square

Lemma 6. *If X, A , and B are finite L -structures such that $A \subseteq B$ and $X \cap B = \emptyset$, then $\delta(X/A) \geq \delta(X/B)$.*

Proof. $\delta((X \cup A) \cup B) \leq \delta(X \cup A) + \delta(B) - \delta((X \cup A) \cap B)$, which simplifies to $\delta(X \cup B) - \delta(B) \leq \delta(X \cup A) - \delta(A)$, as needed. \square

The following two lemmas are from Lemma 2 and the proof of Lemma 3 in [6].

Lemma 7. *Let M be a finite L -structure. Let $A \subseteq M$ and suppose B_j are simply algebraic over A and $A \leq (A \cup \bigcup_j B_j)$, ($j \in J$). Then:*

1. The B_j are pairwise equal or disjoint.
2. $A \cup \bigcup_j B_j$ is a free join of the B_j over A .
3. Suppose $A \subseteq A' \subseteq M$, $A' \leq A' \cup B_j$, and B_j is not a subset of A' ($j=1,2$). Then any isomorphism of B_1 with B_2 over A extends to an isomorphism over A' . In fact, $A' \cup B_j$ is a free join of A' and B_j over A .

Definition 8. *Let \mathcal{C}_0 be the class of finite L -structures C such that if $A \subseteq C \in \mathcal{C}_0$, then $\delta(A) \geq 0$.*

Lemma 9. *Suppose $A, B_1, B_2 \in \mathcal{C}_0$, $A = B_1 \cap B_2$, and $A \leq B_1$. Let E be the free-join of B_1 with B_2 over A . Suppose C^1, \dots, C^r, F are disjoint substructures of E such that each C^i is minimally simply algebraic over F and the structures C^i and C^j are isomorphic over F for each $1 \leq i, j \leq r$. Then one of the following holds:*

1. One of the C^i is contained in $B_1 \setminus A$ and $F \subseteq A$.
2. The set $F \cup \bigcup_{i=1}^r C^i$ is entirely contained in B_1 or B_2 . Further, in the case that $F \cup \bigcup_{i=1}^r C^i$ is entirely contained in B_1 , one of the C^i is contained in $B_1 \setminus A$.
3. $r \leq \delta(F)$

4. For one C^i , setting $X = (F \cap A) \cup (C^i \cap B_2)$, $\delta(X/X \cap A) < 0$. Further, one of the C^j is contained in $B_1 \setminus A$. (Note that this cannot happen if $A \leq B_2$ by Lemma 5).

Continuing as in the standard Hrushovski construction, we define a bounding function μ . μ will tell us how many copies of a given minimally simply algebraic extension to allow within structures in our amalgamation class. For a pair of finite structures A and B , $\mu(B, A) \in \omega$ will decide this bound. Since L may have infinite signature, it is not first order to bound the number of extensions isomorphic to Y over X . In place of this, we will bound the number of extensions of the “form of Y/X ”.

For any disjoint L -structures $\bar{a}, \bar{b} \subseteq C$, we write $tp_{r.q.f.}(\bar{b}/\bar{a})$ for the set $\{R(\bar{x}_i, \bar{y}_i) \mid R \in L, (\bar{b}_i \bar{a}_i) \subseteq (\bar{b} \cup \bar{a})^3 \setminus \bar{a}^3, i \in \omega, \text{ and } R(\bar{b}_i, \bar{a}_i) \text{ holds}\}$. We call this set the relative quantifier-free type of \bar{b} over \bar{a} . We say two relative quantifier-free types are the same if they are equal after a re-ordering of \bar{b} and a re-ordering of \bar{a} . Thus we can talk about the relative quantifier-free type of the set B over A , and we write $tp_{r.q.f.}(B/A)$.

We write $tp_{q.f.}(X)$ to refer to the quantifier-free type of X .

Definition 10. Let Y and X be finite L -structures such that Y is minimally simply algebraic over X . Let $L_{Y/X}$ be the language generated by $\{R \mid R \in L, \exists \bar{x} \subseteq (B \cup A)^3 \setminus A^3(R(\bar{x}))\}$, ie: the language occurring in $tp_{r.q.f.}(Y/X)$. Suppose B and A are finite L -structures such that $tp_{r.q.f.}(B/A)|_{L_{Y/X}} = tp_{r.q.f.}(Y/X)$ and $tp_{q.f.}(X) = tp_{q.f.}(A)$. Then we say the extension B/A is of the form of Y/X .

Fix a function μ from pairs of L -structures (B, A) with B minimally simply algebraic over A to \mathbb{N} so that μ depends only on the atomic type of the pair (B, A) and $\mu(B, A) \geq |A|$. Further, μ must be such that if Γ is a relative quantifier-free type, then there exists a sub-language L' with a finite sub-signature of L so that if $tp_{r.q.f.}(B/A) = \Gamma = tp_{r.q.f.}(B'/A')$ and $tp_{q.f.}(A)|_{L'} = tp_{q.f.}(A')|_{L'}$ then $\mu(B, A) = \mu(B', A')$.

Now we use μ to bound the number of extensions allowed of the form of a given minimally simply algebraic extension and define our amalgamation class. We may also fix a $b \in \omega$ which will become the dimension of the prime model of the theory we are constructing.

Definition 11. Let $\mathcal{C} = \mathcal{C}_{\mu, b}$ be the class of finite L -structures C such that the following hold:

- If $A \subseteq C$ then $\delta(A) \geq \min(|A|, b)$.
- Let Y/X be a minimally simply algebraic extension. Let $B_i, i = 1, \dots, n$, and A be disjoint subsets of C such that B_i/A is an extension of the form Y/X for each i . Then $n \leq \mu(Y, X)$.

Note that if we choose $b \geq \text{arity}(R)$ then realizations of R do not appear in members of \mathcal{C} , thus will not appear in the amalgam we construct. Thus with our choice of a ternary language, only $b < 3$ will yield interesting models. In fact, we will use $b = 2$. This class \mathcal{C} is the class of finite L -structures which we will amalgamate together to form our generic model. To do so, we must show that \mathcal{C} satisfies an amalgamation property. Recall that the free-join of X with Y over $X \cap Y$ is the structure on $X \cup Y$ where a relation holding on \bar{z} implies that \bar{z} is contained entirely in X or entirely in Y .

Lemma 12. (Algebraic Amalgamation Lemma) Suppose $A, B_1, B_2 \in \mathcal{C}$, $B_1 \cap B_2 = A$, and $B_1 \setminus A$ is simply algebraic over A . Let E be the free-join of B_1 and B_2 over A . Then $E \in \mathcal{C}$ unless one of the following holds:

- $B_1 \setminus A$ is minimally simply algebraic over $F \subseteq A$, and B_2 contains $\mu(B_1 \setminus A, F)$ many disjoint extensions of F of the form $(B_1 \setminus A)/F$.
- There exist a set $X \subseteq B_2$ and a subset \hat{L} of $L_{B_1 \setminus A/A}$ such that $(A \cap X)|_{\hat{L}} \not\leq X|_{\hat{L}}$. Further, $B_1|_{\hat{L}}$ contains an isomorphic copy of $X|_{\hat{L}}$.

Proof. We focus on the last condition for E to be in \mathcal{C} . Suppose Y is minimally simply algebraic over X and E contains disjoint sets C^1, \dots, C^r, F where each of the C^j over F is of the form of Y over X . We will look at the structure $E|_{L_{Y/X}}$. Call this structure E' . Our focus will be on using the structure E' to count the C^j s, so we abuse notation and write C^j also for the subset of E' . Here, each of the C^j over F are minimally simply algebraic extensions. Note further that E' is the free-join of $B_1|_{L_{Y/X}}$ with $B_2|_{L_{Y/X}}$ over $A|_{L_{Y/X}}$, both $B_1|_{L_{Y/X}}$ and $B_2|_{L_{Y/X}}$ are in \mathcal{C}_0 , each of the C^j are minimally simply algebraic over F , and $A|_{L_{Y/X}} \leq B_1|_{L_{Y/X}}$. By Lemma 9, there are 4 cases to consider:

1. One of the C^i is contained in $B_1 \setminus A$ and $F \subseteq A$. Since $B_1 \setminus A$ is simply algebraic over A in E , $C^i = B_1 \setminus A$. As the C^j and F are disjoint and one is $B_1 \setminus A$, each of the other C^j as well as F are contained in B_2 . If $r > \mu(Y, X)$ then there must be $\mu(Y, X)$ of them contained in B_2 putting us in the case of the first exception to this lemma.
2. $F \cup \bigcup_{i=1}^r C^i$ is entirely contained in either B_1 or B_2 . Then $r \leq \mu(Y, X)$ as $B_1, B_2 \in \mathcal{C}$.
3. $r \leq \delta(F)$. Then $r \leq \delta(F) \leq |F| = |X| \leq \mu(Y, X)$.
4. For one C^i , setting $X = (F \cap A) \cup (C^i \cap B_2)$, we see that $\delta(X/X \cap A) < 0$. Further, one of the C^j is contained in $B_1 \setminus A$. Letting $\hat{L} = L_{Y/X}$ yields the second exception in our lemma.

□

Lemma 13. (*Strong Amalgamation Lemma*) Suppose $A, B, C \in \mathcal{C}$, $A = B \cap C$, $A \leq B$, and $A \leq C$. Then there exist $E \in \mathcal{C}$ and embeddings $f : B \rightarrow E$, $g : C \rightarrow E$ so that $f|_A = g|_A$, $f(B) \leq E$, and $g(C) \leq E$.

Proof. This is a consequence of inductively applying the previous lemma as in Lemma 4 of [6]. □

Repeatedly applying the strong amalgamation lemma yields a model \mathcal{M} . \mathcal{M} is strongly minimal, saturated, and recursive if μ is recursively approximable from below, as will be shown in the next section.

3 The Amalgam

Using the Strong Amalgamation Lemma, we get a model \mathcal{M} which satisfies the following:

1. \mathcal{M} is countable
2. Every finite substructure of \mathcal{M} is an element of \mathcal{C}
3. Suppose $B \leq \mathcal{M}$, $B \leq C$, and $C \in \mathcal{C}$. Then there exists an embedding $f : C \rightarrow \mathcal{M}$ such that $f|_B = id_B$, and $f(C) \leq \mathcal{M}$.

Note that for any finite $A \subset \mathcal{M}$, there exists a finite $B \subset \mathcal{M}$ such that $A \subseteq B \leq \mathcal{M}$. By a standard back-and-forth argument using finite strong substructures, (1-3) defines \mathcal{M} up to isomorphism.

We would like to show that \mathcal{M} is saturated by showing that any countable elementary supermodel of \mathcal{M} must also have properties (1-3) and thus would have to be isomorphic to \mathcal{M} . The problem is that 3 is not a first-order property. To make the argument work, we replace 3 by $3'$ and $3''$:

$3'$. \mathcal{M} contains an infinite set I such that there are no relations holding on I , and any finite $A \subseteq I$ has the property that $A \leq \mathcal{M}$.

$3''$. Suppose $B \subseteq \mathcal{M}$, $B \leq C$, $C \in \mathcal{C}$, and $C \setminus B$ is simply algebraic over B , say minimally simply algebraic over $F \subseteq B$. Suppose also that for any subset \hat{L} of $L_{C \setminus B/B}$ and any subset X of C there is no set X' such that $X|_{\hat{L}} \cong X'|_{\hat{L}}$ and $(B \cap X')|_{\hat{L}} \not\leq X'|_{\hat{L}}$. Then there are $\mu(C \setminus B, F)$ many disjoint sets A in \mathcal{M} such that A/F is an extension of the form $(C \setminus B)/F$.

Note that $3''$ is defined by a first order schema as each formula involves only a finite language.

Claim 14. $(1, 2, 3)$ and $(1, 2, 3', 3'')$ are equivalent.

Proof. \rightarrow : $3'$ follows from 3 directly and $3''$ follows from the Algebraic Amalgamation Lemma.

\leftarrow : 3 follows as in the proof of the Strong Amalgamation Lemma. □

Corollary 15. \mathcal{M} is saturated.

Proof. Any countable elementary extension of \mathcal{M} satisfies $(1, 2, 3', 3'')$, hence is isomorphic to \mathcal{M} . It follows that there are only countably many types realized in elementary extensions of \mathcal{M} . Hence, there is a saturated countable elementary extension of \mathcal{M} , which \mathcal{M} must be isomorphic to. □

We want to characterize algebraicity in \mathcal{M} . We define $d(A) = \min\{\delta(C) \mid A \subseteq C \subseteq \mathcal{M}, C \text{ finite}\}$. Clearly for any A and x , either $d(xA) = d(A)$ or $d(xA) = d(A) + 1$.

Lemma 16. *If $d(xA) = d(A) + 1$ and $d(yA) = d(A) + 1$, then $(\mathcal{M}, Ax) \cong (\mathcal{M}, Ay)$.*

Proof. Let B be such that $A \subseteq B$, $\delta(B) = d(A)$. Then $B \leq \mathcal{M}$. $d(xB) = d(xA) = d(A) + 1$. Thus $xB \leq \mathcal{M}$, and similarly $yB \leq \mathcal{M}$. Using property 3 and a standard back-and forth argument, we see that (\mathcal{M}, xB) and (\mathcal{M}, yB) are isomorphic. □

We have shown that there is a unique 1-type over A of an element x such that $d(xA) > d(A)$. Next we show that $d(xA) = d(A)$ implies that $x \in \text{acl}_{\mathcal{M}}(A)$.

Lemma 17. *If $d(xA) = d(A)$, then $x \in \text{acl}_{\mathcal{M}}(A)$.*

Proof. Suppose $d(xA) = d(A)$. First, let B be a minimal set such that $A \subseteq B$ and $\delta(B) = d(A)$. We show that B is algebraic over A in \mathcal{M} . Suppose there were two realizations of the type of B over A . Call the second realization B' . Then $\delta(B \cup B') \leq \delta(B) + \delta(B') - \delta(B \cap B') < \delta(B') = d(A)$. The strict inequality is due to B being a minimal set with the properties that $A \subseteq B$ and $\delta(B) = d(A)$. This inequality contradicts the definition of $d(A)$.

Fix E to be a set such that $xA \subseteq E$ and $\delta(E) = d(A)$. Then $\delta(E \cup B) \leq \delta(E) + \delta(B) - \delta(E \cap B)$. If E does not contain B , then $\delta(E \cap B) > d(A)$ by minimality of B . Then

$\delta(E \cup B) \leq d(A) + d(A) - \delta(E \cap B) < d(A)$, again a contradiction. Thus, E contains B and $d(xB) = d(B)$.

Take a sequence of extensions $B_0, B_1, B_2, \dots, B_n$ such that $B_0 = B$, $B_n = E$, and B_{i+1} is a minimal set such that $B_i \subsetneq B_{i+1} \subseteq E$ and $\delta(B_{i+1}) = d(A)$. Then B_{i+1} is a simply algebraic extension of B_i . Thus B_{i+1} is algebraic over B_i . We conclude that E is algebraic over A . In particular, $x \in \text{acl}_{\mathcal{M}}(A)$. \square

Corollary 18. *$\text{Th}(\mathcal{M})$ is strongly minimal.*

Proof. In the previous lemma, we showed that over any set there is a unique non-algebraic 1-type realized in \mathcal{M} . Since \mathcal{M} is saturated, we see that $\text{Th}(\mathcal{M})$ is strongly minimal. \square

Corollary 19. *Let $A \subset \mathcal{M}$ be such that $|A| < b$. Then $\text{acl}_{\mathcal{M}}(A) = A$. In particular, the dimension of the prime model of $\text{Th}(\mathcal{M})$ is b .*

Proof. Suppose $c \in \text{acl}_{\mathcal{M}}(A) \setminus A$. By the characterization of algebraicity above, this means that there exists a set D so that $cA \subseteq D$ and $\delta(D) = d(A) \leq \delta(A) \leq |A|$, but $|A| < |D|$ and $|A| < b$. Thus $\delta(D) < \min(|D|, b)$ yielding a contradiction.

Let $B \leq \mathcal{M}$ be of size at least b so that $\delta(B) = |B|$. Suppose towards a contradiction that $\text{acl}_{\mathcal{M}}(B)$ is finite. By the characterization of algebraicity above, $\delta(\text{acl}_{\mathcal{M}}(B)) = |B|$, so $\text{acl}_{\mathcal{M}}(B) \leq \mathcal{M}$. Let $C \in \mathcal{C}$ be any minimally simply algebraic extension over $\text{acl}_{\mathcal{M}}(B)$, which exists since $|\text{acl}_{\mathcal{M}}(B)| \geq b^1$. Since $\text{acl}_{\mathcal{M}}(B) \leq M$, property 3 yields that C embeds in \mathcal{M} over $\text{acl}_{\mathcal{M}}(B)$. But then the embedded image of C is algebraic over B but not in $\text{acl}_{\mathcal{M}}(B)$, a contradiction. Therefore $\text{acl}_{\mathcal{M}}(B)$ is infinite, yielding a model of dimension $|B|$. Thus the prime model has dimension exactly b . \square

Thus far we have worked generally, constructing a strongly minimal model from any countable relational language L , bounding function μ , and $b \in \omega$. In what follows, we will fix L and b , as well as describe a function μ relative to any $S \subseteq \mathbb{N}$. We will then fix the set S in section 5 to yield the theorems.

Let S be a subset of \mathbb{N} . We view S as a set of pairs of natural numbers $\langle j, k \rangle$ using a standard pairing function (a recursive bijection from \mathbb{N} to $\mathbb{N} \times \mathbb{N}$). We refer to $\{m \in S \mid \exists k (m = \langle j, k \rangle)\}$ as the j^{th} column of S and will write $S^{[j]}$ to denote this set. From the set S , we define the set T to consist of the first two elements of each column not contained in S , ie: $T = \{\langle j, k \rangle \mid \langle j, k \rangle \notin S \text{ and } \neg \exists^2 k' (k' < k \wedge \langle j, k' \rangle \notin S)\}$.

We define L to be the language with signature $\{R\} \cup \{R_i \mid i \in \omega\}$ where each relation symbol is ternary, and we fix $b = 2$.

We enumerate recursively the relative quantifier-free types of extensions Y over X such that Y is minimally simply algebraic over X , $|X| = 3$, and the only relation symbol occurring in $\text{tp}_{r.q.f.}(Y/X)$ is R . We refer to the i^{th} such enumerated relative quantifier-free type as Λ_i .

We say B/A is a Λ_i -extension if $\text{tp}_{r.q.f.}(B/A) \upharpoonright_{\{R\}} = \Lambda_i$. Note that a minimally simply algebraic extension can be a Λ_i -extension and have a relation hold on the base. We now define the bounding function μ .

Definition 20.

$$\mu(Y, X) = \begin{cases} 4 & \text{if } Y/X \text{ is a } \Lambda_{\langle i, k \rangle}\text{-extension, } \langle i, k \rangle \in T, \text{ and } R_i(X) \text{ holds} \\ 4 & \text{if } Y/X \text{ is a } \Lambda_{\langle i, k \rangle}\text{-extension and } \langle i, k \rangle \in S \\ |X| & \text{otherwise} \end{cases}$$

¹For a ternary language, a set $\{a_1, \dots, a_n\}$ has a minimally simply algebraic extension $C = \{c_1, \dots, c_n\}$ formed by setting $R(a_i, c_i, c_{i+1})$ to hold where $i+1$ is $\pmod n$. In any relational language it is also true that sets of size $\geq b$ have minimally simply algebraic extensions.

Note that in the first two cases $|X| = 3$ as the Λ 's are relative quantifier-free types over 3 element sets. Any integer greater than 3 could be used in place of 4 in the above definition.

Lemma 21. *Suppose S is Σ_1 . Then μ recursively approximable from below, and thus \mathcal{M} is a recursively presentable structure.*

Proof. We fix recursive approximations S_s to S such that $S_s \subseteq S_{s+1}$. We will use these recursive approximations to S to build recursive approximations to the amalgamation class and will be able to amalgamate to build \mathcal{M} . We define T_s to be the set comprised of the least 2 elements in each of the first s columns of $\omega \setminus S_s$.

At stage s , define a recursive approximation to μ by

$$\mu_s(Y, X) = \begin{cases} 4 & \text{if } Y/X \text{ is a } \Lambda_{\langle i, k \rangle}\text{-extension, } \langle i, k \rangle \in T_s, \text{ and } R_i(X) \text{ holds} \\ 4 & \text{if } Y/X \text{ is a } \Lambda_{\langle i, k \rangle}\text{-extension and } \langle i, k \rangle \in S_s \\ |X| & \text{otherwise} \end{cases}$$

Note that in the first two cases, $|X| = 3$, as the Λ 's are over 3 element sets. The 4 here corresponds to the same number in the definition of the true μ . We define \mathcal{C}_s , the amalgamation class allowed at stage s , from μ_s .

Let \mathcal{C}_s be the class of finite L -structures C such that the following hold:

- If $A \subseteq C$ then $\delta(A) \geq \min(|A|, 2)$.
- Let Y/X be a minimally simply algebraic extension. Let B_i , $i = 1, \dots, n$, and A be disjoint subsets of C such that B_i/A is an extension of the form Y/X for each i . Then $n \leq \mu_s(Y, X)$.

Since $\mu_s(Y, X) \leq \mu_{s+1}(Y, X)$, we see that $\mathcal{C}_s \subseteq \mathcal{C}_{s+1}$. As $\lim_s \mu_s = \mu$, we see that $\mathcal{C} = \bigcup_s \mathcal{C}_s$. To construct \mathcal{M} , we work in stages. At the s^{th} stage, we amalgamate the first s possible amalgamations allowed in \mathcal{C}_s . As $\mathcal{C} = \bigcup_s \mathcal{C}_s$, every possible amalgamation in \mathcal{C} is amalgamated at a finite stage, and since $\mathcal{C}_s \subseteq \mathcal{C}$, we never leave the amalgamation class \mathcal{C} . This constructs a generic model for \mathcal{C} which is therefore isomorphic to \mathcal{M} . \square

From here forward we assume S is a Σ_1 set, and thus the result of the lemma holds. We fix a recursive presentation of \mathcal{M} , and we refer to this particular presentation as \mathcal{M} from here on.

4 The Restricted Language

To obscure the recursion theoretic content of the construction from the presentation of the model, we will restrict to the language generated by the single relation symbol R . Also, to force the prime model to be recursive in Theorem 1, we will name constants which will identify the prime model.

We fix a non-algebraic pair of elements x and y from \mathcal{M} . By the characterization above, $acl_{\mathcal{M}}(\{x, y\})$ is a Σ_1 set (ie: $z \in acl_{\mathcal{M}}(\{x, y\})$ if and only if $d(\{x, y, z\}) = 2$ if and only if $\exists A \supseteq \{x, y, z\}(\delta(A) = 2)$, which is a Σ_1 condition). Using this observation, we fix a recursive enumeration of $acl_{\mathcal{M}}(\{x, y\})$, $i \mapsto z_i$.

Definition 22. *Let \mathcal{M}' be the model obtained by restricting \mathcal{M} to the language generated by $\{R\}$.*

Let \mathcal{M}'' be the model constructed by adding constant symbols $\{c_i | i \in \omega\}$ to \mathcal{M}' where c_i names the element z_i .

Our next goal is to understand algebraicity in the model \mathcal{M}' . In particular, we will see that the relations that count are R and the R_i such that $S^{[i]} \neq \omega^{[i]}$. From here forth, we call the language generated by $\{R\} \cup \{R_i | S^{[i]} \neq \omega^{[i]}\}$ by the name L' . We will see that the relations in L' are precisely the ones determining algebraicity in \mathcal{M}' . We will construct S so that $S^{[i]}$ is either finite or equals $\omega^{[i]}$, though the results of this section hold even where $T^{[i]}$ contains exactly 1 element.

Recall that $\Lambda_{\langle i,k \rangle}$ is a relative quantifier-free type of an extension of a 3-element set and involves only the relation symbol R . In the context of a first order formula, we write $\Lambda_{\langle i,k \rangle}(\bar{y}, \bar{x})$ to represent the formula which states that \bar{y} over \bar{x} is a $\Lambda_{\langle i,k \rangle}$ -extension. Note that $\Lambda_{\langle i,k \rangle}(\bar{y}, \bar{x})$ is a first order formula involving only the relation symbol R .

Lemma 23. *Let i be an integer such that $S^{[i]} \neq \omega^{[i]}$, and let $\langle i, k \rangle$ be an element of $T^{[i]}$. Then $\mathcal{M} \models \forall \bar{x} (R_i(\bar{x}) \leftrightarrow \exists^4 \bar{y} \Lambda_{\langle i,k \rangle}(\bar{y}, \bar{x}))$*

Proof. \leftarrow : If there are 4 disjoint $\Lambda_{\langle i,k \rangle}$ -extensions over \bar{x} and $R_i(\bar{x})$ does not hold, then taking the finite set A comprised of the 4 extensions and \bar{x} , we see that $A \notin \mathcal{C}$ as this explicitly violates the μ -bound. This contradicts property (2) of \mathcal{M} .

\rightarrow : Suppose $R_i(\bar{x})$ holds. Then $\delta(\bar{x}) = 2$, which shows $\bar{x} \leq \mathcal{M}$. By (3''), we see that there are 4 disjoint $\Lambda_{\langle i,k \rangle}$ -extensions over \bar{x} . \square

Since each of the relations $R_i \in L'$ are definable in \mathcal{M}' , we will abuse notation and say $R_i(\bar{x})$ holds in \mathcal{M}' to mean that the equivalent statement involving only R holds in \mathcal{M}' . Similarly for \mathcal{M}'' . This shows that these relations still count in the reduct \mathcal{M}' . Lemma 27 shows that these are the only relations that still count.

Lemma 24. *Suppose $tp_{r.q.f.}(B/A) = tp_{r.q.f.}(Y/X)$ and B/A is a minimally simply algebraic extension. Then Y/X is a minimally simply algebraic extension.*

Proof. It is easy to check that $\delta(Y/X) = 0$ and $\delta(Y_0/X) > 0$ for any $\emptyset \neq Y_0 \subsetneq Y$, as these are true for B/A . Thus Y is simply algebraic over X . Also, each $x \in X$ satisfies some relation in $Y \cup X$ with an element in Y , as it must be so for B/A . Thus if $Z \subsetneq X$, then $\delta(Y/Z) > \delta(Y/X) = 0$, so Y is minimally simply algebraic over X . \square

Definition 25. *For finite $A \subseteq M'$, let $\delta'(A) = \delta(A|_{L'}) = |A| - |R(A)| - \sum_{R_j \in L'} |R_j(A)|$.*

Let $d'(A) = \min\{\delta'(B) | A \subseteq B \subseteq M', B \text{ finite}\}$.

Let $\mathcal{C}_{L'}$ be the class of finite L' -structures in \mathcal{C} .

Lemma 26. *If $A \in \mathcal{C}$, then $A|_{L'} \in \mathcal{C}_{L'}$.*

Proof. For any $X' \subseteq A|_{L'}$, let X be the expansion to L . Then $\delta(X') \geq \delta(X) \geq \min(|X|, 2) = \min(|X'|, 2)$.

Let Y' be a minimally simply algebraic extension over X' . Suppose C'_1, \dots, C'_n, F' are disjoint subsets of $A|_{L'}$ where each C'_i over F' is an extension of the form of Y' over X' . Note that $L_{Y'/X'} \subseteq L'$. Let C_1, \dots, C_n, F be the expansions to L . Let Y over X be an extension so that $X \cong F$ and $tp_{r.q.f.}(Y/X) = tp_{r.q.f.}(Y'/X')$. By Lemma 24 Y is minimally simply algebraic over X . Each C_i over F is of the form of Y over X and $tp_{q.f.}(X)|_{L'} = tp_{q.f.}(X')$. Since $A \in \mathcal{C}$, $n \leq \mu(Y, X)$. Observe from the definition of μ that $\mu(Y, X)$ depends only on the quantifier-free L' -type of $Y \cup X$. Thus $n \leq \mu(Y, X) = \mu(Y', X')$. \square

Lemma 27. *$\mathcal{M}|_{L'}$ is generic for the class $\mathcal{C}_{L'}$.*

Proof. We need to show that $\mathcal{M}|_{L'}$ satisfies the conditions to be a generic model of $\mathcal{C}_{L'}$. We use the versions of (1, 2, 3', 3'') for $\mathcal{C}_{L'}$:

1: $\mathcal{M}|_{L'}$ is countable

2: For any finite $A \subseteq \mathcal{M}|_{L'}$, $A \in \mathcal{C}_{L'}$

3': $\mathcal{M}|_{L'}$ contains an infinite set I such that there are no relations holding on I , and any finite $A \subseteq I$ has the property that $A \leq \mathcal{M}|_{L'}$.

3'': Suppose $B \subseteq \mathcal{M}|_{L'}$, $B \leq C$, $C \in \mathcal{C}_{L'}$, and $C \setminus B$ is simply algebraic over B , say minimally simply algebraic over $F \subseteq B$. Suppose also that for any subset \hat{L} of $L_{C \setminus B/B}$ and any $X \subseteq C$, there is no set X' such that $X|_{\hat{L}} \cong X'|_{\hat{L}}$ and $(B \cap X')|_{\hat{L}} \not\leq X'|_{\hat{L}}$. Then there are $\mu(C \setminus B, F)$ many disjoint sets A in \mathcal{M} such that A/F is an extension of the form $(C \setminus B)/F$.

1 is equivalent to 1 above. 2 follows immediately from 2 above and the previous lemma.

3' follows from 3' above since $B \leq \mathcal{M}$ implies that $B|_{L'} \leq \mathcal{M}|_{L'}$.

Given $A \subseteq \mathcal{M}$ such that $F \subseteq B = A|_{L'}$ and $C \in \mathcal{C}_{L'}$ are as in 3'', we set D to be the L -structure so that $tp_{r.q.f.}(D/A) = tp_{r.q.f.}(C/B)$. Let E be the expansion of F to L . By Lemma 24, $D \setminus A$ is simply algebraic over A and minimally simply algebraic over E . Since $L_{D \setminus A/A} = L_{C \setminus B/B} \subseteq L'$ and $(D, A, E)|_{L'} \cong (C, B, F)$, we see that (D, A, E) satisfies the hypothesis of 3'' for \mathcal{M} . Applying 3'' for \mathcal{M} to the extension of D over A , we have $\mu(D \setminus A, E)$ disjoint extensions of the form of $(D \setminus A)/E$ over E in \mathcal{M} . Since μ depends only on the quantifier-free L' -type, in $\mathcal{M}|_{L'}$ this gives $\mu(C \setminus B, F)$ disjoint extensions of the form of $(C \setminus B)/F$ over F . \square

Lemma 28. $x \in acl_{\mathcal{M}'}(A)$ if and only if $d'(xA) = d'(A)$.

Proof. Above we showed that for \mathcal{M} the generic model of \mathcal{C} , algebraicity meant $d(xA) = d(A)$. By the analogous argument for $\mathcal{M}|_{L'}$, we see that algebraicity here means $d'(xA) = d'(A)$. Since $\mathcal{M}|_{L'}$ is a definitional expansion of \mathcal{M}' , algebraicity is the same for \mathcal{M}' . \square

Lemma 29. \mathcal{M}' and \mathcal{M}'' are both recursive, saturated, and strongly minimal.

Proof. \mathcal{M}' is recursive, saturated, and strongly minimal, as it is a reduct to a recursive language of a model with all of these properties.

\mathcal{M}'' is recursive since the assignment of the constants is recursive. It is strongly minimal, as adding constants to a strongly minimal theory retains strong minimality. Take I an infinite algebraically independent sequence in \mathcal{M} beginning with $\{x, y\}$. $I - \{x, y\}$ is algebraically independent over the algebraic closure of $\{x, y\}$ in \mathcal{M} . Thus $I - \{x, y\}$ is algebraically independent in \mathcal{M}'' . This shows that \mathcal{M}'' has infinite algebraic dimension, thus is saturated. \square

5 Defining S

Thus far we have constructed the two models \mathcal{M}' and \mathcal{M}'' relative to any given Σ_1 set S . We aim for a construction where $SRM(Th(\mathcal{M}')) = \{\omega\}$ and $SRM(Th(\mathcal{M}'')) = \{0, \omega\}$. To ensure this, we need to diagonalize against the possible finite-dimensional models of each theory. In this section, we construct the Σ_1 set S to ensure these results.

We want to ensure that the finite dimensional models of $Th(\mathcal{M}')$ and $Th(\mathcal{M}'')$ are not recursive. There is no 0-dimensional model of $Th(\mathcal{M}')$ (ie: $acl(\emptyset) = \emptyset$), so we will diagonalize only against positive dimensional models. We fix a recursive enumeration of all pairs (f, U) where f is a partial recursive function from the set of quantifier-free formulas in the language $\{R\} \cup \{c_i | i \in \omega\} \cup \mathbb{N}$ to $\{\text{true}, \text{false}\}$ and U is a non-empty finite

subset of \mathbb{N} . Note that if f gives the quantifier-free diagram of a model, $\text{dom}(f)$ contains symbols for the elements of the model, which is why we include the symbols from \mathbb{N} . This is to be interpreted as f giving the quantifier-free diagram of a model N with universe \mathbb{N} and U representing a basis of the model. Note that even though we work with functions f describing structures in the language including constants, the construction allows for functions which give a model of $\text{Th}(M')$, ie: where each c_i is not interpreted. Thus our construction of S will simultaneously diagonalize against finite-dimensional models of $\text{Th}(M')$ and of $\text{Th}(M'')$.

We will describe a routine for enumerating S . For the i^{th} pair (f, U) , we will have an i^{th} subroutine Routine_i whose job it is to ensure that this pair does not represent a model N with a basis U satisfying either of the theories of M' or M'' .

Given a pair (f, U) , at stages we read off information about the model it describes from f_s , the computation of f at stage s . We let N_0 be the empty model, and let N_s be comprised of all $n \leq s$ such that for each $m < n$, $f_s(n = m) \downarrow = \text{'false'}$. In N_s , we say $R(\bar{x})$ holds if $f_s(R(\bar{x})) \downarrow = \text{'true'}$. We say for $R_i \in L$ with $i < s$, $R_i(\bar{x})$ holds if there is an $\langle i, k \rangle \in T_s^{[i]}$ such that $N_s \models \exists^4 \bar{y} \Lambda_{\langle i, k \rangle}(\bar{y}, \bar{x})$, where $\Lambda_{\langle i, k \rangle}(\bar{y}, \bar{x})$ has already been defined in N_s as a conjunction of R -statements. For a set A of natural numbers, we write $\delta_s(A)$ for $\delta(A)$ as A is seen in the structure N_s . Finally, we set $K_s \subseteq N_s$ to be the set of elements $x \in N_s$ such that $f_s(x = c_i) \downarrow = \text{'true'}$ for some $i \leq s$.

Routine_i is the only part of our program allowed to enumerate anything into $S^{[i]}$. When Routine_i is initialized, $S^{[i]} = \emptyset$. The routine runs in parts as follows:

Part 1) Wait until a stage s when there is some set $X \subseteq N_s$ and a set $K \subseteq K_s$ such that $(X \cup U \cup K)|_{R_i}$ is a minimally simply algebraic extension over $(U \cup K)|_{R_i}$. Once found, for the duration of its run Routine_i refers to these sets as X and K .

Part 2) The first thing Routine_i does when it reaches part 2 is to define the set of obstructions to moving to part 3. Suppose we reach part 2 on stage t . A set $Y \subseteq N_t$ is an obstruction to moving to part 3 if $\delta_t(Y/K_t) < |U|$ and $U \subseteq Y$. For each $k \in \omega$, if during a stage $s > t$ an element is enumerated into $S^{[k]}$, then we say R_k is removed. If at a stage s enough R_k are removed so that counting only the non-removed R_k , $\delta_s(Y/K_t) \geq |U|$, then we say the obstruction Y has been removed. That is, if

$$\left[|Y \cup K_t| - |R(Y \cup K_t)| - \sum_{\substack{R_j \text{ not} \\ \text{removed}}} |R_j(Y \cup K_t)| \right] - \left[|K_t| - |R(K_t)| - \sum_{\substack{R_j \text{ not} \\ \text{removed}}} |R_j(K_t)| \right] \geq |U|,$$

then the obstruction Y is removed.

If for each tuple $\bar{x} \in X \cup U \cup K$, $N_s \models \exists^4 \bar{y} \Lambda_{\langle i, l \rangle}(\bar{y}, \bar{x}) \leftrightarrow \exists^4 \bar{y} \Lambda_{\langle i, m \rangle}(\bar{y}, \bar{x})$ where $\{\langle i, l \rangle, \langle i, m \rangle\} = T_s^{[i]}$, then we say Routine_i is ready for part 3. If Routine_i is ready for part 3 and all obstructions have been removed, Routine_i moves to part 3.

Part 3) Routine_i takes the least element of $\omega^{[i]}$ which has not yet been enumerated into S and enumerates it into S . Routine_i then goes back to part 2.

The possible outcomes of a run of Routine_i are that it gets stuck in part 1, it gets stuck in part 2, or it cycles between part 2 and part 3 infinitely often. In the first case, $S^{[i]} = \emptyset$. In the second case, $S^{[i]}$ is a finite initial segment of $\omega^{[i]}$, and in the third case $S^{[i]} = \omega^{[i]}$. In any case, we will show that either N does not satisfy the right theory or U is not its basis.

6 The Main Theorems

In the previous section we defined a Σ_1 set S , and in the section before we gave a construction of two models \mathcal{M}' and \mathcal{M}'' from any fixed Σ_1 set. We fix \mathcal{M}' and \mathcal{M}'' to be those models obtained by applying the construction to the set S defined in the previous section.

It is clear that $\omega \in SRM(Th(\mathcal{M}'))$ and $0, \omega \in SRM(Th(\mathcal{M}''))$. The first is because \mathcal{M}' has a recursive presentation and is saturated. The second is because \mathcal{M}'' has a recursive presentation and is saturated and the set of constants in \mathcal{M}'' is algebraically closed and infinite, hence also a model of the same theory. Since the set of constants is Σ_1 in the recursive presentation of \mathcal{M}'' , they form a recursive prime model. It remains to show that for any other $n \in \omega + 1$, n is not in $SRM(Th(\mathcal{M}'))$ or $SRM(Th(\mathcal{M}''))$.

Theorem 30. $SRM(Th(\mathcal{M}')) = \{\omega\}$

Proof. Suppose N is a recursive model of $Th(\mathcal{M}')$, and N has a finite basis U . Let i be the index of the pair (f, U) where f is the recursive function describing the quantifier-free diagram of N . Note that we let the domain of f include symbols for constants, but $f(c_i = n) \downarrow = \text{'false'}$ for each $n \in \omega$.

Case 1: *Routine* _{i} gets stuck in part 1.

$R_i \in L'$, as *Routine* _{i} is never in stage 3. Note that $Th(\mathcal{M}')$ has no model with a basis of size < 2 . So, we may assume $|U| \geq 2$. We are guaranteed by $3''$ that if N satisfies $Th(\mathcal{M}')$ then N must contain minimally simply algebraic extensions involving only the relation R_i over U . Thus $N \not\models Th(\mathcal{M}')$

Case 2: *Routine* _{i} gets stuck in part 2.

Case 2a: *Routine* _{i} gets stuck in part 2 because it is never ready for part 3.

This means that for some $\bar{x} \in X \cup U$ and $\langle i, l \rangle, \langle i, m \rangle \in T_s^{[i]}$, $N \not\models \exists^4 \bar{y} \Lambda_{\langle i, l \rangle}(\bar{y}, \bar{x}) \leftrightarrow \exists^4 \bar{y} \Lambda_{\langle i, m \rangle}(\bar{y}, \bar{x})$. Since *Routine* _{i} never gets to part 3 again, $T^{[i]} = T_s^{[i]}$. By Lemma 23, $\mathcal{M}' \models \exists^4 \bar{y} \Lambda_{\langle i, l \rangle}(\bar{y}, \bar{x}) \leftrightarrow R_i(\bar{x}) \leftrightarrow \exists^4 \bar{y} \Lambda_{\langle i, m \rangle}(\bar{y}, \bar{x})$. Thus $N \not\models Th(\mathcal{M}')$.

Case 2b: There is an obstruction Y which is never removed.

As N is assumed to be a model of $Th(\mathcal{M}')$, for each $n \in \omega$ and constant symbol c_i , $N \not\models n = c_i$. Thus $K_t = \emptyset$ for each stage t , and we can simplify notation and write $\delta_s(Y)$ for $\delta_s(Y/K_t)$. When counting the non-removed relations, $\delta_s(Y) = \delta_s(Y/K_t) < |U|$. Since the obstruction is never removed, $\delta'(Y) < |U|$, contradicting U being an independent set in N .

Case 3: *Routine* _{i} loops through part 2 and part 3 infinitely often.

The construction is built so that in this case, U will not be a basis for the model N . We will derive a contradiction from the assumption that U is a basis for N . From this assumption, we see that X is algebraic over U , which means that there is a set Y such that $\delta'(Y) = |U|$, and $X \cup U \subseteq Y$. Let t be a stage when *Routine* _{i} enters part 2 which is large enough that $Y \subseteq N_t$ and for each relation R_j in L' occurring on Y , $S_t^{[j]} = S^{[j]}$. Such a t exists since by construction $S^{[j]}$ is finite for each $R_j \in L'$. We will show that *Routine* _{i} never enters part 3 after stage t , leading to a contradiction.

As *Routine* _{i} is ready for part 3 each time it leaves part 2, we see that at all stages s after *Routine* _{i} completes part 1, N_s realizes occurrences of R_i on X . Let $s > t$ be a stage when *Routine* _{i} is in part 2. Then $\delta_s(Y) < \delta'(Y)$ since R_i occurs on X but does not count in δ' . Then R_i is a relation which has not been removed since entering part 2 and neither have any of the relations counted in δ' , so counting only the non-removed relations, $\delta_s(Y) < |U|$. Thus, Y is an obstruction which is never removed after stage t , contradicting our being in case 3.

In any case, we get a contradiction to the assumption that N is a recursive model of $Th(\mathcal{M}')$ with finite basis U . \square

Lemma 31. *Let U be a finite subset of \mathcal{M}'' . Then $x \in acl_{\mathcal{M}''}(U)$ if and only if there is a finite set C of elements named by constants and $x \in acl_{\mathcal{M}'}(U \cup C)$.*

Proof. The right-to-left direction is trivial. To prove the left-to-right direction, take an algebraic formula $\phi(x, U, C)$ defining x over U involving constants C . See that ϕ is an algebraic formula over $U \cup C$ in \mathcal{M}' . \square

Theorem 32. $SRM(Th(\mathcal{M}'')) = \{0, \omega\}$

Proof. Suppose N is a recursive model of $Th(\mathcal{M}'')$ and N has a finite basis U . Let i be the index of the pair (f, U) where f is the function describing the quantifier-free diagram of N .

Case 1: *Routine_i* gets stuck in part 1.

$R_i \in L'$, as *Routine_i* is never in stage 3. \mathcal{M}' has minimally simply algebraic extensions involving only the relation R_i . Thus N does not satisfy $Th(\mathcal{M}')$. In particular, if $|U| \geq 2$ then $3''$ guarantees that there are minimally simply algebraic extensions of U in N involving only the relation symbol R_i . If $|U| = 1$, then $3''$ guarantees the same for $U \cup \{c_k\}$ for any constant c_k .

Case 2: *Routine_i* gets stuck in part 2.

Case 2a: *Routine_i* gets stuck in part 2 because it is never ready for part 3.

This means that for some $\bar{x} \in X \cup U \cup K$, $\langle i, l \rangle, \langle i, m \rangle \in T_s^{[i]}$, $N \not\models \exists^4 \bar{y} \Lambda_{\langle i, l \rangle}(\bar{y}, \bar{x}) \leftrightarrow \exists^4 \bar{y} \Lambda_{\langle i, m \rangle}(\bar{y}, \bar{x})$. But since *Routine_i* never gets to part 3 again, $T^{[i]} = T_s^{[i]}$. By Lemma 23, $\mathcal{M}' \models \exists^4 \bar{y} \Lambda_{\langle i, l \rangle}(\bar{y}, \bar{x}) \leftrightarrow R_i(\bar{x}) \leftrightarrow \exists^4 \bar{y} \Lambda_{\langle i, m \rangle}(\bar{y}, \bar{x})$. Thus $N \not\models Th(\mathcal{M}')$.

Case 2b: There is an obstruction Y that is never removed.

There is a finite set of constants C in N such that counting only the non-removed relations on Y , $\delta_s(Y/C) < |U|$. As the obstruction is never removed, $\delta'(Y/C) < |U|$, implying that U is not an independent set over the constants C . Hence U is not algebraically independent over \emptyset .

Case 3: *Routine_i* loops through part 2 and part 3 infinitely often.

The construction is built so that in this case, U will not be a basis for the model N . We will derive a contradiction from the assumption that U is a basis for N . From this assumption, we see that X is algebraic over U , which means that there is a finite set Y and a finite set of constants $C \supseteq K$ such that $\delta'(Y/C) = |U|$ and $X \cup U \subseteq Y$. Let t be a stage when *Routine_i* enters part 2 which is large enough that $Y \cup C \subseteq N_t$ and for each relation R_j in L' occurring on $Y \cup C$, $S_t^{[j]} = S^{[j]}$. Such a t exists as $S^{[j]}$ is finite for each $R_j \in L'$. We will show that *Routine_i* never enters part 3 after stage t , leading to a contradiction.

As *Routine_i* is ready for part 3 each time it leaves part 2, we see that at all stages s after *Routine_i* completes part 1, N_s realizes occurrences of R_i on X . Let $s > t$ be a stage when *Routine_i* is in part 2. Then $\delta_s(Y/C) < \delta'(Y/C)$ since R_i occurs on X but does not count in δ' . Then R_i is a relation which has not been removed since entering part 2 and neither have any of the relations counted in δ' , so counting only the non-removed relations, $\delta_s(Y/C) < |U|$. Thus, Y is an obstruction which is never removed after stage t , contradicting our being in case 3.

In any case, we get a contradiction to the assumption of N being a recursive model of $Th(\mathcal{M}'')$ with finite basis U . \square

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