

THE BOREL COMPLEXITY OF THE CLASS OF MODELS OF FIRST-ORDER THEORIES

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ABSTRACT. We investigate the descriptive complexity of the set of models of first-order theories. Using classical results of Knight and Solovay, we give a sharp condition for complete theories to have a $\mathbf{\Pi}_\omega^0$ -complete set of models. We also give sharp conditions for theories to have a $\mathbf{\Pi}_n^0$ -complete set of models. Finally, we determine the Turing degrees needed to witness the completeness.

1. INTRODUCTION

We characterize the possible Borel complexities of the set of models of a first-order theory. For a single formula φ , Wadge [Wa83, I.F.3 and I.F.4], using a result by Keisler [Ke65], showed that if φ is an \exists_n -formula which is not equivalent to a \forall_n -formula, then the set of models of φ is a Σ_n^0 -complete set under Wadge reduction. We extend this result to considering (possibly incomplete) first-order theories T and giving conditions on T determining the complexity of $\text{Mod}(T)$, the set of models of T .

We show that a complete theory T has no \forall_n -axiomatization for any finite n if and only if $\text{Mod}(T)$ is $\mathbf{\Pi}_\omega^0$ -complete. Prior to this result, showing that $\text{Mod}(T)$ is $\mathbf{\Pi}_\omega^0$ -complete was difficult even for familiar theories, e.g., Rossegger [Ro20] asked this for the theory TA of true arithmetic. We also show that for any finite n , a (possibly incomplete) theory T has a \forall_n -axiomatization if and only if $\text{Mod}(T)$ is $\mathbf{\Pi}_n^0$. If T does not have a \forall_n -axiomatization, then T is Σ_n^0 -hard.

By Vaught's proof [Va74] of the Lopez-Escobar theorem, showing that the set of models of T is Σ_n^0 (or $\mathbf{\Pi}_n^0$) is equivalent to showing that T is equivalent to a Σ_n^{in} -formula (or $\mathbf{\Pi}_n^{\text{in}}$ -formula, respectively). Also, Wadge's argument shows that if the set of models of T is Σ_n^0 and not $\mathbf{\Pi}_n^0$, then it must be Σ_n^0 -complete. Thus, an equivalent way to present our main results is in terms of when a first-order theory T is equivalent to a formula in $L_{\omega_1\omega}$. For example, it follows that a first-order theory is equivalent to a $\mathbf{\Pi}_n^{\text{in}}$ -sentence if and only if it has a \forall_n -axiomatization.

2020 *Mathematics Subject Classification.* 03C62,03C52,03E15.

Key words and phrases. models of arithmetic, Borel complexity, countable models, descriptive set theory.

The third author's research was partially supported by AMS-Simons Foundation Collaboration Grant 626304. The fourth author's work was supported by the European Union's Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie grant agreement No. 101026834 — ACOSE. Part of this research was carried out while the third author was visiting Technical University of Vienna as a guest professor, and he would like to express his gratitude to this university. – The authors would like to express their gratitude to Roman Kossak for connecting them with Ali Enayat and Albert Visser, who were able to establish complementary results to ours on sequential theories.

This is related to Keisler's result [Ke65] that was recently reproved by Harrison-Trainor and Kretschmer [HK23] that if a first-order formula is definable by an infinitary Π_n^{in} -formula, then it must already be definable by a \forall_n -formula. In fact, our result applied to a single formula implies this via an easy application of compactness. This result shows that, though infinitary logic can express much more than finitary logic, it cannot express things more efficiently, i.e., in fewer quantifiers, than finitary logic. We do note that Keisler's result holds not only for formulas equivalent to Π_n^{in} -formulas in $L_{\omega_1\omega}$, but also for formulas in admissible fragments of $L_{\infty\omega}$, whereas our proof does not yield this generality.

Interestingly, all three proofs are quite different. Keisler used games and saturated models, Harrison-Trainor and Kretschmer used arithmetical forcing, and we use iterated priority constructions. One advantage of our technique is that, while combinatorially quite complicated, the metamathematics involved are quite tame. This suggests that our results are a consequence of, if not equivalent to, compactness. By contrast, Wadge's result that $\text{Mod}(\varphi)$ is Σ_n^0 -complete if φ is \exists_n and not equivalent to a \forall_n -formula relies upon Borel determinacy. A second advantage is that we can determine the exact complexity of an oracle needed to compute the continuous reduction witnessing that $\text{Mod}(\varphi)$ is Σ_n^0 -complete. Finally, the main advantage is that we consider theories, not simply formulas.

2. PRELIMINARIES

Given a Polish space X , the Borel hierarchy on X gives us a way to stratify subsets of X in terms of their descriptive complexity. A natural space is the space of countably infinite structures in a countable relational vocabulary τ which we can view as a closed subspace of $2^{\mathbb{N}}$ as follows. Fix an enumeration of the atomic τ -formulas $(\varphi_i(x_0, \dots, x_i))_{i \in \mathbb{N}}$; then given a τ -structure \mathcal{A} with universe \mathbb{N} , define its *atomic diagram* by

$$D(\mathcal{A})(i) = \begin{cases} 1 & \mathcal{A} \models \varphi_i[x_j \mapsto j : j \leq i], \\ 0 & \text{otherwise.} \end{cases}$$

and let $\text{Mod}(\tau) \subseteq 2^{\mathbb{N}}$ be the set of atomic diagrams of τ -structures with universe \mathbb{N} . Then it is easy to see that $\text{Mod}(\tau)$ is a closed subset of Cantor space and thus a Polish space via the subspace topology.

For a first-order theory T , $\text{Mod}(T) = \{D(\mathcal{A}) : \mathcal{A} \models T\}$ is a canonical subset of $\text{Mod}(\tau)$, and it is natural to ask how complex $\text{Mod}(T)$ is in terms of its Borel complexity. It is not hard to see that $\text{Mod}(T)$ can be at most Π_ω^0 : It follows from Vaught's proof [Va74] of the Lopez-Escobar theorem [Lo65] that an isomorphism-invariant subset of $\text{Mod}(\tau)$ is Π_α^0 if and only if it is definable by a Π_α^{in} -formula in the infinitary logic $L_{\omega_1\omega}$ for $\alpha < \omega_1$. Note that we use the notation Π_α^{in} to refer to formulas in $L_{\omega_1\omega}$ at that complexity, and we use \forall_n to refer to $L_{\omega\omega}$ -formulas with n quantifier blocks beginning with a \forall . Since $\text{Mod}(T)$ is the set of models of the infinitary formula $\bigwedge_{\varphi \in T} \varphi$, and every φ is \exists_n for some $n \in \mathbb{N}$, we get that $\text{Mod}(T)$ is at most Π_ω^0 .

However, for a fixed theory T , it turns out to be quite difficult to establish that $\text{Mod}(T)$ is not simpler. The main theorem of this paper establishes a complete characterization of first-order theories T where $\text{Mod}(T)$ is Π_ω^0 -complete. To establish this notion of completeness, we use Wadge reducibility; a subset X_1 of a Polish space Y_1 is *Wadge reducible* a subset X_2 of a Polish space Y_2 (denoted as

$X_1 \leq_W X_2$) if there is a continuous function $f : Y_1 \rightarrow Y_2$ such that for all $y \in Y_1$, $y \in X_1$ if and only if $f(y) \in X_2$. A set $X \subseteq Y$ is Γ -hard for a pointclass Γ if for every $Z \in \Gamma$, $Z \leq_W X$, and X is Γ -complete if $X \in \Gamma$ and X is Γ -hard. We are also able to provide sharp bounds on the Turing degree of the continuous function witnessing the reduction. Wadge [Wa83] showed that there is only one *Wadge degree* of a Σ_α^0 -set which is not Π_α^0 and vice versa. We refer the reader to Kechris [Ke95] for more on Wadge reducibility and a thorough introduction to descriptive set theory.

3. THEORIES WITHOUT BOUNDED AXIOMATIZATION

Definition 3.1. A first-order theory T is *boundedly axiomatizable* if there is some n so that T has a \forall_n -axiomatization.

Our main result for theories that are not boundedly axiomatizable is the following

Theorem 3.2. *A complete first-order theory T has a Π_ω^0 -complete set of models if and only if T is not boundedly axiomatizable.*

In fact, Theorem 3.2 follows directly from the following more technical fact.

Theorem 3.3. *Let T be any complete first-order theory for which there is a collection of complete theories $\{T_n\}_{n \in \omega}$ such that for all $n \in \omega$, $T \neq T_n$ but $T \cap \exists_n = T_n \cap \exists_n$. Then the collection of models of T is Π_ω^0 -complete. Indeed, for each Π_ω^0 -set P , there is a continuous function mapping any $p \in P$ to a model of T , and any $p \notin P$ to a model satisfying T_n for some n .*

We show in the next example the necessity of the assumption of completeness for the theory T in Theorem 3.2.

Example 3.4. Let \mathcal{L}_k be disjoint relational languages, and for each k , let φ_k be an \mathcal{L}_k -sentence which is \exists_k and not equivalent to any \forall_k -sentence. Let \mathcal{L} be $\bigcup_k \mathcal{L}_k \cup \{R_i \mid i \in \omega\}$, where each R_i is unary. Let T say that the set of realizations of each R_i is disjoint, and at most one is non-empty. Further, let T say that any relation from \mathcal{L}_k can only hold on tuples from the set of realizations of R_k . Let T further say that if R_k is non-empty, then φ_k holds. It is direct to see that T has no \forall_n -axiomatization for any n and yet $\text{Mod}(T)$ is Σ_ω^0 .

We defer the proof of Theorem 3.3 to Section 5. Here we state some corollaries of Theorem 3.2:

Corollary 3.5. *For any completion T of Peano arithmetic PA, in particular for true arithmetic, the set of models of T is Π_ω^0 -complete. \square*

This follows from an observation of Rabin [Ra61] (which he suspects to have been known before) that no consistent extension of PA can be boundedly axiomatizable.

While Example 3.4 shows that Theorem 3.2 cannot be generalized to hold for incomplete theories, for many incomplete theories, one can use Theorem 3.3 to get a similar result. One simply has to find a suitable completion T and suitable theories T_n . One example of such a theory is PA.

Corollary 3.6. *Peano arithmetic has a Π_ω^0 -complete set of models.*

Proof. Let $T = \text{TA}$, and let T_n be a consistent completion of $\text{TA} \cap \exists_n$, where $\text{B}\Sigma_n^0$, the bounding principle for Σ_n^0 -formulas, fails. That such T_n exists for every n follows from a result by Parsons [Pa70], see also [Ka91, Theorem 10.4]. Using Theorem 3.3 with this T and $(T_n)_{n \in \omega}$, we get that $\text{Mod}(\text{PA})$ is Π_ω^0 -complete. \square

Answering our question about other theories of arithmetic, Enayat and Visser [EVta] showed that no complete sequential theory can be boundedly axiomatizable. The sequential theories were first defined by Pudlák [Pu83] and rephrased by Pakhomov and Visser [PV22] as follows:

Definition 3.7. Given a theory T , we denote by $\text{AS}(T)$ (*adjunctive set theory*) the extension of T by a new binary relation symbol \in and the axioms

- AS1: $\exists x \forall y (y \notin x)$, and
- AS2: $\forall x \forall y \exists z \forall u (u \in z \leftrightarrow (u \in x \vee u = y))$.

A theory is *sequential* if it allows a definitional extension to $\text{AS}(T)$.

Note here that Adjunctive Set Theory does not even require Extensionality. Pudlák’s original definition was in terms of being able to define Gödel’s β -function, which then allows for a weak coding of sequences. (Note that any extension of a sequential theory is again sequential.) Examples of sequential theories include PA^- (by Jeřábek [Je12]) and essentially all versions of set theory, but not Robinson’s Q (by Visser [Vi17]). Thus Enayat and Visser’s result [EVta] yields that $\text{Mod}(T)$ is Π_ω^0 -complete for essentially any “foundational” complete theory, in particular, any completion of PA^- .

Finally, note that our reduction in Theorem 3.3 produces models of different theories T_n for different x in the Σ_ω^0 -outcome, depending on how we witness that $x \notin P$. We note that it is necessary to use infinitely many theories in the Σ_ω^0 -outcome, since the union of $\text{Mod}(T_i)$ for finitely many theories T_i is always Π_ω^0 , and we are reducing a Σ_ω^0 -hard set.

4. THEORIES WITH BOUNDED AXIOMATIZATION

In this section, we will present results on the Wadge degrees of models of first-order theories with bounded axiomatization via the quantifier complexity of their axiomatizations. Our proofs will rely on the following lemma that will rely on theorems of Knight and Solovay. We delay its proof to Section 5.

Lemma 4.1. *Suppose $n \geq 1$ and T^+ and T^- are distinct complete theories such that $T^- \cap \exists_n \subseteq T^+ \cap \exists_n$. Then for any $P \in \Sigma_n^0$, there is a Wadge reduction f such that $f(p) \in \text{Mod}(T^+)$ if $p \in P$, and $f(p) \in \text{Mod}(T^-)$ otherwise. In particular, $\text{Mod}(T^+)$ is Σ_n^0 -hard, and $\text{Mod}(T^-)$ is Π_n^0 -hard.*

In order to apply Lemma 4.1 to incomplete theories, we use the following Lemma which allows us to find completions satisfying the hypotheses of Lemma 4.1.

Definition 4.2. A *level-sentence set* for \mathcal{L} is either the set of \exists_n - or the set of \forall_n -sentences in \mathcal{L} for some n .

For a level-sentence set Λ , we let $\neg\Lambda$ be the set of sentences equivalent to the negation of a sentence in Λ .

Lemma 4.3. *Let Λ be a level-sentence set for \mathcal{L} . Let A be a set of finitary sentences and φ a finitary sentence such that $A \not\vdash \varphi \leftrightarrow \psi$ for any $\psi \in \neg\Lambda$. Then there are complete consistent theories $T^+ \supseteq A \cup \{\varphi\}$ and $T^- \supseteq A \cup \{\neg\varphi\}$ such that $\text{Th}_\Lambda(T^-) \subseteq \text{Th}_\Lambda(T^+)$. Furthermore, if T is any theory consistent with $A \cup \{\varphi\} \cup \text{Th}_\Lambda(A \cup \{\neg\varphi\})$, then T^+ can be chosen to contain T .*

Proof. The lemma follows from the following two claims that allow us to choose such T^+ and T^- .

Claim 4.4. The theory $A \cup \{\varphi\} \cup \text{Th}_\Lambda(A \cup \{\neg\varphi\})$ is consistent.

Proof. Suppose that $A \cup \{\varphi\} \cup \text{Th}_\Lambda(A \cup \{\neg\varphi\})$ is inconsistent. By compactness, there is $\psi \in \text{Th}_\Lambda(A \cup \{\neg\varphi\})$ such that $A \cup \{\varphi\} \vdash \neg\psi$. But then $A \vdash \varphi \leftrightarrow \neg\psi$ as well as $\neg\psi \in \neg\Lambda$, a contradiction. \square

Now, choose T^+ to be any complete extension of $A \cup \{\varphi\} \cup \text{Th}_\Lambda(A \cup \{\neg\varphi\})$. Observe that if T is any theory consistent with $A \cup \{\varphi\} \cup \text{Th}_\Lambda(A \cup \{\neg\varphi\})$, then T^+ can be chosen to contain T .

Claim 4.5. The theory $\text{Th}_{-\Lambda}(T^+) \cup A \cup \{\neg\varphi\}$ is consistent.

Proof. Suppose not, then by compactness, there is $\psi \in \text{Th}_{-\Lambda}(T^+)$ such that $A \cup \{\neg\varphi\} \vdash \neg\psi$. But then $\neg\psi \in \text{Th}_\Lambda(A \cup \{\neg\varphi\}) \subseteq T^+$, contradicting that T^+ is consistent. \square

Let T^- be a completion of $\text{Th}_{-\Lambda}(T^+) \cup \{\neg\varphi\}$. Observe that T^- and T^+ satisfy the lemma. \square

Corollary 4.6. *Let Λ be a level-sentence set, and let T be a theory which is not Λ -axiomatizable (i.e., $\text{Th}_\Lambda(T)$ does not imply all of T). Then there are complete theories T_0, T_1 such that $T \subseteq T_0$, T is inconsistent with T_1 , and $\text{Th}_\Lambda(T_0) \subseteq \text{Th}_\Lambda(T_1)$.*

Proof. Let $A = \text{Th}_\Lambda(T)$, and let $\varphi \in T$ be so that $A \not\vdash \varphi$. Observe that $A \not\vdash \varphi \leftrightarrow \psi$ for any $\psi \in \Lambda$, since otherwise ψ would be in $\text{Th}_\Lambda(T) = A$, contradicting $A \not\vdash \varphi$.

Observe also that $T \cup \text{Th}_{-\Lambda}(A \cup \{\neg\varphi\})$ is consistent. Otherwise, there would be a formula $\psi \in \Lambda$ so that $T \vdash \psi$, thus $\psi \in A$, and $A \cup \{\neg\varphi\} \models \neg\psi$. But then $A \vdash \varphi$, which is a contradiction. So, we can apply Lemma 4.3 to the triple $\neg\Lambda, A, \varphi$ to get two complete theories $T^- \supseteq A \cup \{\neg\varphi\}$ and $T^+ \supseteq T$ with $\text{Th}_{-\Lambda}(T^-) \subseteq \text{Th}_{-\Lambda}(T^+)$. Finally, let $T_0 = T^+$ and $T_1 = T^-$. \square

Lemma 4.7. *Let T be a theory without a \forall_n -axiomatization. Then $\text{Mod}(T)$ is Σ_n^0 -hard.*

Proof. By Corollary 4.6, we have complete theories $T_0 \supseteq T$ and T_1 inconsistent with T so that $\text{Th}_{\forall_n}(T_0) \subseteq \text{Th}_{\forall_n}(T_1)$. Thus $\text{Th}_{\exists_n}(T_1) \subseteq \text{Th}_{\exists_n}(T_0)$, and applying Lemma 4.1 shows that $\text{Mod}(T)$ is Σ_n^0 -hard. \square

Lemma 4.8. *Let T be a theory without an \exists_n -axiomatization. Then $\text{Mod}(T)$ is Π_n^0 -hard.*

Proof. By Corollary 4.6, we have complete theories $T_0 \supseteq T$ and T_1 inconsistent with T so that $\text{Th}_{\exists_n}(T_0) \subseteq \text{Th}_{\exists_n}(T_1)$, and applying Lemma 4.1 shows that $\text{Mod}(T)$ is Π_n^0 -hard. \square

Theorem 4.9. *Let T be a theory and $n \in \omega$. Then $\text{Mod}(T) \in \Pi_n^0$ if and only if T is \forall_n -axiomatizable.*

Proof. If $\text{Mod}(T) \in \Pi_n^0$, then it is not Σ_n^0 -hard. So, by Lemma 4.7 it must have a \forall_n -axiomatization. On the other hand, if T is \forall_n -axiomatizable, then $\text{Mod}(T) \in \Pi_n^0$, as the infinitary conjunction over all sentences in the axiomatization is Π_n^{in} . \square

For \exists_n -axiomatizable theories, the situation is not as simple as the one for \forall_n -axiomatizable theories seen in Theorem 4.9. If A is a \exists_n -axiomatization of T , then $\text{Mod}(T) = \text{Mod}(\psi)$, where $\psi = \bigwedge_{\varphi \in A} \varphi$. However, ψ will in general not be $\Sigma_\alpha^{\text{in}}$,

but rather $\Pi_{\alpha+1}^{\text{in}}$. Combining this with the contrapositive of Lemma 4.8, we can get the following

Proposition 4.10. *Let T be a theory and $n \in \omega$. If $\text{Mod}(T) \in \Sigma_n^0$, then T is \exists_n -axiomatizable. On the other hand, if T is \exists_n -axiomatizable, then $\text{Mod}(T) \in \Pi_{n+1}^0$. \square*

We now give examples of \exists_n -axiomatizable theories of different Wadge degrees showing that the bounds in Proposition 4.10 cannot be improved.

Example 4.11. For $k \leq 2$, there are \exists_k -axiomatizable \aleph_0 -categorical theories of Wadge degree Π_{k+1}^0 .

Proof. For $k = 1$, let τ be the signature consisting of one unary relation symbol P , and let T be the theory saying that P is infinite and coinfinite. T is easily seen to be \exists_1 -axiomatizable, \aleph_0 -categorical, and $\text{Mod}(T)$ is Π_2^0 -complete.

For $k = 2$, let τ be the signature consisting of a single binary relation symbol R . Let T say that R is symmetric and for every x , there is at most one y so $R(x, y)$. Finally, let T say that there are infinitely many x satisfying $\exists y R(x, y)$ and infinitely many x satisfying $\neg \exists y R(x, y)$. Then T is \aleph_0 -categorical, \exists_2 -axiomatizable, and $\text{Mod}(T)$ is Π_3^0 -complete. \square

Example 4.12. There is a finitely \exists_3 -axiomatizable \aleph_0 -categorical theory of Wadge degree Σ_3^0 .

Proof. Consider the theory of the linear ordering $2 \cdot \mathbb{Q} + 1 + \mathbb{Q}$ together with its successor relation S . This theory is \aleph_0 -categorical and is axiomatizable by the axioms for linear orderings, the definition of the successor relation, and the following \exists_3 -formula:

$$(\star) \quad \exists x [(\forall y < x)(\exists z(S(y, z) \vee S(z, y)) \wedge (\exists u S(y, u) \rightarrow \forall v \neg S(v, y))) \wedge (\forall y > x)(\forall u > x)(\neg S(y, u) \wedge \neg S(u, y)) \wedge \exists z z > x]$$

One can easily verify that $L \cong 2 \cdot \mathbb{Q} + 1 + \mathbb{Q}$ for any countable linear ordering L satisfying (\star) , thus this is a finite \exists_3 -axiomatization for an \aleph_0 -categorical theory. It was shown in [GRta, Theorem 3.3] that the isomorphism class of $2 \cdot \mathbb{Q} + 1 + \mathbb{Q}$ is Σ_4^0 -complete in the space of linear orderings (without successor relation). Now, towards a contradiction, assume it is not Σ_3^0 -hard in $\text{Mod}(\{\leq, S\})$. Then Lemma 4.7 gives a $\Pi_3^{\text{in}}(\leq, S)$ -sentence φ such that $\text{Mod}(\varphi) = \text{Iso}(2 \cdot \mathbb{Q} + 1 + \mathbb{Q}, \leq, S)$. But clearly φ translates into a $\Pi_4^{\text{in}}(\leq)$ -formula, contradicting that $\text{Iso}(2 \cdot \mathbb{Q} + 1 + \mathbb{Q}, \leq)$ is Σ_4^0 -complete in $\text{Mod}(\{\leq\})$. \square

We next show that 3 is minimal possible in Example 4.12. This is similar to a result of Arnold Miller [Mi83] that states that no countable structure can have a Σ_2^0 -isomorphism class.

Proposition 4.13. *Let φ be a consistent Σ_2^{in} -sentence. Then φ has a finitely generated model. In particular, if T is a complete relational theory and $\text{Mod}(T) \in \Sigma_2^0$, then $\text{Mod}(T) = \emptyset$.*

Proof. Suppose φ is Σ_2^{in} , i.e., of the form $\bigvee_{i \in \omega} \exists \bar{x} \theta_i(\bar{x})$, where θ_i is a conjunction of \forall_1^0 -sentences. Assume without loss of generality that $(\mathcal{A}, \bar{a}) \models \theta_i(\bar{a})$ for some i , then every substructure of (\mathcal{A}, \bar{a}) satisfies $\theta_i(\bar{a})$ and thus the substructure of \mathcal{A} generated by \bar{a} satisfies φ .

Now assume that T is a complete relational theory and $\text{Mod}(T) \in \Sigma_2^0$. Then by Lopez-Escobar, $\text{Mod}(T) = \text{Mod}(\varphi)$ for a Σ_2^{in} -formula φ . It then follows from the above argument that T has a finite model \mathcal{A} . Hence $\text{Mod}(T) = \text{Iso}(\mathcal{A})$ by the completeness of T . Hence, T does not have a countably infinite model and $\text{Mod}(T)$ is empty. \square

Next we show that Examples 4.11 and 4.12 can be generalized to higher quantifier levels.

Lemma 4.14. *Let $n \geq 2$. Let T be an \exists_n -axiomatizable theory so that $\text{Mod}(T)$ is Σ_n^0 -complete (or Π_{n+1}^0 -complete, respectively). Let T' be the Δ_2^0 -Marker extension of T (see [AM, Lemma 2.8]). Then T' is an \exists_{n+1} -axiomatizable theory so that $\text{Mod}(T')$ is Σ_{n+1}^0 -complete (or Π_{n+2}^0 -complete, respectively).*

Proof. We focus on the case where $\text{Mod}(T)$ is Σ_n^0 -complete, with the Π_{n+1}^0 -complete case being similar. Note that $\text{Mod}(T')$ is Σ_{n+1}^0 , since from a structure B , it is $\Delta_3^0(B)$ to check that it is a Δ_2^0 -Marker extension of a structure \hat{B} , with \hat{B} being uniformly $\Delta_2^0(B)$. Finally, B is a model of T' if and only if \hat{B} is a model of T , which is $\Sigma_n^0(B')$. Putting all together, we get that $\text{Mod}(T')$ is Σ_{n+1}^0 .

Since $\text{Mod}(T)$ is Σ_n^0 -complete, there is a continuous reduction of $P_n = \{k \frown p \mid k \in p^{(n)}, p \in 2^\omega\}$ to $\text{Mod}(T)$. We will convert this into a continuous reduction of $P_{n+1} = \{k \frown p \mid k \in p^{(n+1)}, p \in 2^\omega\}$ to $\text{Mod}(T')$. Since P_{n+1} is a Σ_{n+1}^0 -complete subset of ω^ω , this shows $\text{Mod}(T')$ is Σ_{n+1}^0 -complete.¹

Let g be the continuous map reducing P_n to $\text{Mod}(T)$, and let D be an oracle which computes g . Given $k \frown p$, $g(k \frown p')$ is uniformly computable from $D \oplus p'$. Then $D \oplus p$ can uniformly compute a copy of the Δ_2^0 -Marker extension of $g(k \frown p')$. This yields the Σ_{n+1}^0 -hardness of $\text{Mod}(T')$.

It is straightforward to check that if T is \exists_n -axiomatizable for $n \geq 2$, then T' is \exists_{n+1} -axiomatizable. \square

Since Marker extensions preserve \aleph_0 -categoricity and finite axiomatizability, we generalize Examples 4.11 and 4.12 to higher quantifier levels.

Example 4.15. For every $n \geq 1$, there is an \exists_n -axiomatizable \aleph_0 -categorical theory T so that $\text{Mod}(T)$ is of Wadge degree Π_{n+1}^0 .

For every $n \geq 3$, there is a finitely \exists_n -axiomatizable \aleph_0 -categorical theory T so that $\text{Mod}(T)$ is of Wadge degree Σ_n^0 .

Example 4.16. For any $n \geq 3$, there is an \exists_n -axiomatizable \aleph_0 -categorical theory T so that $\text{Mod}(T)$ is a properly Δ_{n+1}^0 -set.

Proof. Fix T_0 to be an \exists_{n-1} -axiomatizable \aleph_0 -categorical theory so that $\text{Mod}(T_0)$ is Π_n^0 -complete. Fix T_1 to be an \exists_n -axiomatizable \aleph_0 -categorical theory so that $\text{Mod}(T_1)$ is Σ_n^0 -complete. Let T have a unary predicate U and say that the set of elements realizing U is a model of T_0 and the set of elements realizing $\neg U$ is a model of T_1 . Then T is \exists_n -axiomatizable and $\text{Mod}(T)$ is $D_2(\Sigma_n^0)$ -complete. \square

¹To see that P_{n+1} is Σ_{n+1}^0 -complete, first note that it is Σ_{n+1}^0 . If there was D such that P_{n+1} is $\Pi_{n+1}^0(D)$, then P_{n+1} would be $\Delta_{n+1}^0(D)$. Hence, we would get that for any C computing D that $n \in C^{(n+1)}$ if and only if $n \frown \chi_C \in P_{n+1}$ and this would be $\Delta_{n+1}^0(C)$, hence computable from $C^{(n)}$. But this would contradict that the Turing jump is proper. So, by Wadge's lemma, P_{n+1} is Σ_{n+1}^0 -complete.

We observe that a special case of Theorem 4.9 implies a case of a theorem of Keisler [Ke65, Corollary 3.4], recently reproved by Harrison-Trainor and Kretschmer [HK23].

Theorem 4.17. *If a finitary first-order formula φ is equivalent to $\psi \in \Pi_n^{\text{in}}$, then there is a \forall_n -formula θ such that $\varphi \equiv \theta$.*

Proof. By adding constants, we may assume that φ is a sentence. Since φ is equivalent to ψ , we get that $\text{Mod}(\{\varphi\}) \in \mathbf{\Pi}_n^0$. Thus Theorem 4.9 shows that φ has a \forall_n -axiomatization. Compactness implies that φ is equivalent to a single \forall_n -sentence. \square

Combining our results from this section we get the following characterization.

Theorem 4.18. *Let T be a theory and $n \in \omega$. Then the following are equivalent.*

- (1) *T has a \forall_n -axiomatization but no \forall_{n-1} -axiomatization.*
- (2) *The Wadge degree of $\text{Mod}(T)$ is in $[\Sigma_{n-1}^0, \Pi_n^0]$.*

Note that the intervals $[\Sigma_{n-1}^0, \Pi_n^0]$ contain \aleph_1 -many different Δ_n^0 Wadge degrees.

Question 4.19. Which Δ_n^0 -Wadge degrees are the degree of $\text{Mod}(T)$ for some (complete) finitary first-order theory?

5. PROOFS OF THE TWO TECHNICAL RESULTS

In the present section, we will prove Theorem 3.3 and Lemma 4.1. We will first prove Theorem 3.3 and then introduce some minor modifications to the proof to prove Lemma 4.1.

The proof of Theorem 3.3 relies on theorems of Knight [Kn87] and Solovay (see [Kn99]) Those proofs proceed via worker arguments, a framework for (possibly infinitely) iterated priority constructions. Recently, applications of such systems have been found in descriptive set theory by Marks/Montalbán (in preparation) and Day/Greenberg/Harrison-Trainor/Turetsky [DGHTta].

A *Scott set* \mathcal{S} is a subset of 2^ω that is closed under Turing reducibility, join, and satisfies weak König's lemma, i.e., if $T \in \mathcal{S}$ codes an infinite binary tree, then there is a path f through T such that $f \in \mathcal{S}$. An *enumeration* of a countable Scott set \mathcal{S} is a set $R \in 2^\omega$ satisfying $\mathcal{S} = \{R^{[n]} \mid n \in \omega\}$, i.e., \mathcal{S} equals the set of columns of R . If $A = R^{[i]}$, then we say that i is an R -index for A .

Marker (see Macintyre/Marker [MM84]) showed that if X computes an enumeration R of a Scott set \mathcal{S} , then X also computes an *effective enumeration*, i.e., an enumeration where the closure properties of the Scott set are witnessed by computable functions. In particular, there is a computable function $f(a, b)$ so that if $\varphi_a(R^{[b]})$ is an infinite tree T , then $R^{[f(a, b)]}$ is a path through T .

We are now ready to state Knight's theorem and Solovay's refinement.

Theorem 5.1 ([Kn87, Theorem 1.1]). *Let T be a complete theory. Suppose $R \leq_T X$ is an enumeration of a Scott set \mathcal{S} , and $e \in \omega$ is so that for each n , $\Phi_e^{X^{(n-1)}}$ is an R -index for $T \cap \exists_n$. Then T has a model \mathcal{B} with $\mathcal{B} \leq_T X$.*

Theorem 5.2 ([Kn99, Theorem 2.5]). *Let T be a complete theory. Suppose $R \leq_T X$ is an enumeration of a Scott set \mathcal{S} , with functions t_n which are $\Delta_n^0(X)$ uniformly in n , such that for each n , $\lim_s t_n(s)$ is an R -index for $T \cap \exists_n$, and for all s , $t_n(s)$ is an R -index for a subset of $T \cap \exists_n$. Then T has a model \mathcal{B} with $\mathcal{B} \leq_T X$.*

In fact, Solovay showed that this is sharp:

Theorem 5.3 ([Kn99, Corollary 3.5]). *If T is a completion of PA, the degrees of non-standard models of T are the degrees of sets X which compute enumerations R of a Scott set, equipped with functions t_n which are $\Delta_n^0(X)$ uniformly in n , such that for each n , $\lim_s t_n(s)$ is an R -index for $T \cap \exists_n$, and for all s , $t_n(s)$ is an R -index for a subset of $T \cap \exists_n$.*

5.1. Proof of Theorem 3.3. The proof of Theorem 3.3 has two parts. First, we fix a $\mathbf{\Pi}_\omega^0$ -set P , theories T and $\{T_n\}_{n \in \omega}$, and show how to obtain the Scott set \mathcal{S} and how to produce functionals t_n such that, using an input $p \in 2^\omega$, we satisfy Theorem 5.2 and thus output a model of T if $p \in P$ and a model of some T_n otherwise.

The second part concerns the uniformity of the construction. As stated in the literature, it is not immediately apparent that we can go from the premises to the conclusion in Theorem 5.1 and Theorem 5.2 in a way that is continuous in p . We will thus take a closer look at the worker method to see that these results are uniform.

Part 1. Given a $\mathbf{\Pi}_\omega^0$ -set P , we can fix a decreasing sequence of $\mathbf{\Pi}_n^0$ -sets P_n such that $P = \bigcap_{n \geq 1} P_n$. Since our construction will heavily depend on it, we will describe the Borel codes of P (and of the sets P_n) fairly explicitly: Fix a (computable) basis $\{U_i\}_{i \in \omega}$ for the topology of 2^ω . The Borel code for P_n will be a Borel code (i.e., a tree) C_n of nodes $\sigma \in \omega^{n+1}$ such that each $\tau \in \omega^n$ has an extension in C_n ; the interpretation is that

$$(5.1) \quad P_n = \bigcap_{j_1} \bigcup_{j_2} \cdots \bigcap_{j_n} \bigcup_{\langle j_1 \dots j_n j \rangle \in C_n} U_j$$

if n is odd, and

$$(5.2) \quad P_n = \bigcap_{j_1} \bigcup_{j_2} \cdots \bigcup_{j_n} \bigcap_{\langle j_1 \dots j_n j \rangle \in C_n} \overline{U_j}$$

if n is even (where \overline{U} is the complement of U). We will assume that $P_2 \supseteq P_3 \dots$; and so $P = \bigcap_{n \geq 2} P_n$ has Borel code

$$C = \bigcup_{n \geq 2} (\langle n \rangle \hat{\ } C_n).$$

(Note that the labeling function is implicit in our coding and thus can be suppressed.)

Next, we fix an enumeration R of a Scott set \mathcal{S} containing $T \cap \exists_k$ for each k and $\{T_n \cap \exists_k\}_{n, k \in \omega}$. Let $Y = C \oplus R \oplus T \oplus \bigoplus_{n \in \omega} T_n$. We will describe a computation Φ_e satisfying the hypotheses of Theorem 5.1, namely, that $\Phi_e^{(Y \oplus p)^{(n-1)}}$ is an R -index for $T_p \cap \exists_n$ for a complete theory T_p . Furthermore, we will ensure that if $p \in P$ then $T_p = T$; and if $p \notin P$ then $T_p = T_n$ for some $n \in \omega$. Note that we are applying Theorem 5.1 with the oracle $X = Y \oplus p$.

We describe the index e by giving a uniform method of computing an R -index for $T \cap \exists_n$ from $(Y \oplus p)^{(n-1)}$. For $n = 1$, we output a fixed index for $T \cap \exists_1$. For $n \geq 2$, $\Phi_e^{(Y \oplus p)^{(n-1)}}$ depends on whether $p \in P_{n-1}$. Since membership of p in each basic open set U_i is p -computable, it is easy to verify by induction from (5.1) and (5.2) that membership of p in P_{n-1} is $(C \oplus p)^{(n-1)}$ -computable, given that

checking membership in each infinite union and intersection takes one more jump to decode.

For $n \geq 2$, let Φ_e be the algorithm defined as follows. Given an oracle of the form $(Y \oplus p)^{(n-1)}$, let k_0 be the least $k < n$ such that $p \notin P_k$ if such k exists, and n otherwise. Let $\Phi_e^{(Y \oplus p)^{(n-1)}}(s)$ output the least R -index of $T_{k_0} \cap \exists_n$. Note that finding this index is not effective in R and T_{k_0} but is effective in $(R \oplus T_{k_0})' \leq_T Y' \leq_T (Y \oplus p)^{(n-1)}$ (needing one jump here is why we treat the case $n = 1$ differently).

We have just produced a uniform sequence of computations, thus we can find a single index e (note that Y and p are used in the oracle, but not in identifying the index) and can use Theorem 5.1 to obtain a model \mathcal{M}_p computable from $Y \oplus p$ such that $\mathcal{M}_p \models T$ if $p \in P$, and $\mathcal{M}_p \models T_n$ for some $n \in \omega$ otherwise. It remains to show that we can produce the model \mathcal{M}_p continuously from p .

Part 2. We have to show that there is a continuous map $p \rightarrow \mathcal{M}_p$, where \mathcal{M}_p is the structure produced by Theorem 5.1 for p . We refer to the proof in Knight [Kn87], though these remarks on uniformity apply also to the proof of Solovay's result. Knight presents a worker argument. That is, for each $n \in \omega$, Knight gives (uniformly) a procedure for an algorithm computing a sequence of computations using $X^{(n-1)}$. These computations depend on the computations being given by other oracles, chiefly $X^{(n)}$. There is an application of the recursion theorem to show that there is in fact a single index i so that each worker $X^{(n)}$ can be taken to be doing its computations via the same index $\Phi_i^{X^{(n)}}$, and thus each worker uses the correct approximations to the computations of the other workers.

As the description of what each worker is doing (relative to all the others) is uniform, and the recursion theorem is uniform, the entire construction is uniform. That is, there is computable function $\sigma(e, j)$ so that whenever T, X, e satisfies the hypothesis of the theorem with $R = \Phi_j^X$, then $\sigma(e, j)$ is an index (equal to the i in the previous paragraph – σ comes from the uniformity of the recursion theorem) so that $\Phi_{\sigma(e, j)}^X$ is the atomic diagram of a model \mathcal{B} so that M models the theory T .

5.2. Proof of Lemma 4.1. The proof is almost the same as the proof of Theorem 3.3, except that one of our approximation functions will not be constant, so we use Theorem 5.2. The only difference is in the first step, i.e., how we obtain the sequence of functions t_n to apply this theorem. We use $C \oplus R \oplus p$ as an oracle for computing \mathcal{M}_p , where C is a Borel code for the Σ_n^0 -set P and R is an enumeration of a Scott set which contains both T^+ and T^- . Let $P = \bigcup_{i \in \omega} P_i$, where the P_i are Δ_n^0 . Note that Marker's theorem shows that R can compute another enumeration of the same Scott set which is effective, so we may assume that we can uniformly compute R -indices for $T^+ \cap \exists_n$ and $T^- \cap \exists_n$.

Here, for $k < n$, we let t_k^p be the algorithm that simply outputs the least index i of R such that $R^{[i]} = T^- \cap \exists_k = T^+ \cap \exists_k$. We let t_n^p be the algorithm that checks, given s and using $(p \oplus C)^{(n-1)}$, whether $p \in P_i$ for $t < s$. If not, we output the least R -index for $T^- \cap \exists_n$. If so, we output the least R -index for $T^+ \cap \exists_n$. Note that the value of $t_n^p(s)$ may change at most once as s increases. For $k \geq n$, we let t_k^p be the algorithm that checks using $(p \oplus C)^{(k)}$ whether $p \in P$ and outputs the index for $T^+ \cap \exists_k$ if it is, and the index for $T^- \cap \exists_k$ otherwise. The fact that $T^- \cap \exists_n \subseteq T^+ \cap \exists_n$ guarantees that the premises for Theorem 5.2 are satisfied. The rest of the proof is as in the proof of Theorem 3.3.

6. THE EFFECTIVITY OF THE REDUCTIONS

While the reductions in Theorem 3.2 and Lemma 4.8 are continuous, they are not effective without oracles. In this section, we explore which oracles are necessary. In fact, by proving Theorem 3.2 through Theorem 3.3, we have already taken a non-computable step by choosing the theories T_n . We will see that we can improve the oracle needed for the reduction in Theorem 3.2 by not doing this.

We introduce the following notion to formalize the question of which Turing degrees are necessary to witness the hardness of the class $\text{Mod}(T)$.

Definition 6.1. We say that D witnesses the Γ -hardness of $Y \subseteq 2^\omega$ if for every Borel code C for a set $X \in \Gamma$, there is a Turing operator Φ so that $\Phi^{D \oplus C \oplus p} \in Y$ if and only if $p \in X$ for every $p \in 2^\omega$.

If the index for the Turing operator Φ does not depend on C , then we say D uniformly witnesses the Γ -hardness of Y .

A Turing degree \mathbf{d} (uniformly) witnesses the Γ -hardness of Y if it contains a set D (uniformly) witnessing the Γ -hardness of Y .

Theorem 6.2. Let T be a complete theory which is not boundedly axiomatizable. Suppose $R \leq_T X$ is an enumeration of a Scott set \mathcal{S} , with functions t_n which are $\Delta_n^0(X)$ uniformly in n , such that for each n , $\lim_s t_n(s)$ is an R -index for $T \cap \exists_n$, and for all s , $t_n(s)$ is an R -index for a subset of $T \cap \exists_n$.

Then X uniformly witnesses the $\mathbf{\Pi}_\omega^0$ -hardness of $\text{Mod}(T)$.

Proof. We are given a $\mathbf{\Pi}_\omega^0$ set $P = \bigcap P_n$ with Borel code C so that each P_n is $\mathbf{\Pi}_n^0$ and an X computing the enumeration R . By Marker's theorem [MM84], there is an X -computable effective enumeration \hat{R} of \mathcal{S} and it is $\Delta_2^0(X)$ to transfer R -indices to \hat{R} -indices. Thus, we may assume that R is an effective enumeration of \mathcal{S} .

We now give two sequences of functions $u_n^p(s)$ and $v_n^p(s)$, each uniformly $\Delta_n^0(X \oplus C \oplus p)$. For each p and n , we ensure that $\lim_s u_n^p(s)$ is an R -index for $T_p^- \cap \exists_n$ for a theory T_p^- , and each $u_n^p(s)$ is an R -index for a subset of $T_p^- \cap \exists_n$. Similarly, for each p and n , we ensure that $\lim_s v_n^p(s)$ is an R -index for $T_p^+ \cap \exists_n$ for a theory T_p^+ , and each $v_n^p(s)$ is an R -index for a subset of $T_p^+ \cap \exists_n$. Thus, applying Theorem 5.2, we build continuous maps $\alpha^- : p \mapsto M_p^- \models T_p^-$ and $\alpha^+ : p \mapsto M_p^+ \models T_p^+$, and we will make one of these two reduce P to $\text{Mod}(T)$. Further, we will see that either α^- or α^+ works for every Borel set \hat{P} with code \hat{C} .

We will ensure $u_n^p(s) = v_n^p(s) = t_n(s)$ whenever $p \in P$, so both α^- and α^+ produce models of T when $p \in P$. We will also ensure that either $T_p^- \neq T$ for every p , or $T_p^+ \neq T$ for every p .

The algorithm to determine $u_n^p(s)$ and $v_n^p(s)$ each first finds the least $k < n$ so that $p \notin P_k$. If no such k exists, then $u_n^p(s) = v_n^p(s) = t_n(s)$.

We first describe the algorithm for $u_n^p(s)$: Using our oracle $(X \oplus C \oplus p)^{(n-1)}$, we first compute $\hat{u} = \lim_s u_{n-1}^p(s)$ and $\hat{t} = t_{n-1}(s)$, then we check if $R^{[\hat{u}]} = R^{[\hat{t}]}$. If not, then we have already ensured that the theory $T_p^- \neq T$, and we simply use the effectiveness of the enumeration R to find the R -index of any \exists_n -theory consistent with the \exists_{n-1} -theory $R^{[\hat{u}]}$. If $R^{[\hat{u}]} = R^{[\hat{t}]}$, then let T^* be the $\exists_n \cup \forall_n$ -theory determined by this column. Then we next check if there is any \exists_n -sentence ρ so that $T^* \not\models \rho$ and $T^* \models \neg\rho$. If not, then again there is only one completion, and we let $u_n^p(s) = t_n(s)$. If there is such a ρ , then we use the effectiveness of the

enumeration R to find two R -indices of \exists_n -theories i^- and i^+ so that $\rho \notin R^{[i^-]}$ and $\rho \in R^{[i^+]}$.

We let $u_n^p(s) = i^-$ for all s .

The algorithm for $v_n^p(s)$ is exactly the same, except in the last case where we let $v_n^p(s) = i^+$.

Note that in every case except the last case, we have ensured that $u_n^p(s)$ is a constant function which gives the index for a consistent \exists_n -fragment of a theory which is not equal to $T \cap \exists_n$.

Note that for any k , there exists infinitely many $n > k$ so that there is more than one consistent way to extend $T \cap \exists_n$ to an \exists_{n+1} -theory. Otherwise, T would be boundedly axiomatizable. So in the algorithm computing $u_n^p(s)$ and $v_n^p(s)$, if $p \notin P$, then we are infinitely often in the final case. Thus, for every $p \notin P$, $T_p^- \neq T_p^+$.

We may suppose that there exists a $p \notin P$ so that $T_p^- = T$, as otherwise α^- is our needed reduction. Let k be so that $p \notin P_k$. This means that for every $n > k$, whenever there is more than one extension of $T \cap \exists_n$ to an \exists_{n+1} -theory, we have $T \cap \exists_{n+1} = R^{[i^-]}$.

We will show that α^+ is a reduction. Suppose towards a contradiction that $\hat{p} \notin P_m$ for some m and $T_{\hat{p}}^+ = T$. Let $\ell > k, m$ be so that $T \cap \exists_\ell$ does not imply $T \cap \exists_{\ell+1}$. Then the algorithms for T_p^- and $T_{\hat{p}}^+$ choose different theories on level ℓ , but $T_p^- = T$, so $T_{\hat{p}}^+ \cap \exists_{\ell+1} \neq T \cap \exists_{\ell+1}$ contradicting $T_{\hat{p}}^+ = T$.

Note that which of α^- or α^+ gives a reduction is determined by whether fragments of T are equal to approximations to it which are computed uniformly by X . In other words, the reduction Φ reducing P to $\text{Mod}(T)$ computable from $X \oplus C \oplus p$ is uniform across pairs (P, C) where C is a Borel code for a $\mathbf{\Pi}_\omega^0$ -set P , but there is non-uniformity across X and T . \square

Note that by Theorem 5.3, if T is a completion of PA, a set X satisfies the hypotheses of Theorem 6.2 if and only if it computes a nonstandard model of T . We now observe that this bound is sharp.

Corollary 6.3. *Let T be a completion of PA. If a Turing degree \mathbf{d} computes a non-standard model of T , then \mathbf{d} uniformly witnesses the $\mathbf{\Pi}_\omega^0$ -hardness of $\text{Mod}(T)$.*

Let T be a completion of PA^- . If \mathbf{d} does not compute a non-standard model of T , then it does not witness the Σ_2^0 -hardness of $\text{Mod}(T)$. Moreover, if \mathbf{d} does not compute any model of T , then it does not witness the 2^ω -hardness of $\text{Mod}(T)$.

Proof. Let T be a completion of PA. Let \mathbf{d} compute a non-standard model of T . Then Theorem 5.3 shows that \mathbf{d} satisfies the hypotheses of Theorem 6.2, so \mathbf{d} uniformly witnesses the $\mathbf{\Pi}_\omega^0$ -hardness of $\text{Mod}(T)$.

Let T be a completion of PA^- . Suppose that \mathbf{d} witnesses the Σ_2^0 -hardness of $\text{Mod}(T)$. Let P be the set of $\chi_{\{n\}}$, where $n \in D''$ for some fixed $D \in \mathbf{d}$, and observe P is Σ_2^0 with Borel code $C \in \mathbf{d}$. There is a function Ψ so that $\Psi(p)$ is uniformly computable in $D \oplus p$, and $\Psi(p)$ is a model of T if and only if $p \in P$. If the range of Ψ contains any non-standard model of T , then $D \oplus p$ computes a non-standard model of T for some $p \in P$. But every $p \in P$ is computable, so there is a \mathbf{d} -computable non-standard model of T . The alternative is that $\Psi(p)$ is the standard model of TA whenever $p \in P$. The standard model of TA has a Π_2^1 -Scott sentence, so this causes D'' to be $\Pi_2^0(D)$: $n \in D''$ if and only if $\Psi(\chi_{\{n\}}) \cong (\mathbf{N}, +, \cdot)$, which contradicts the fact that D'' is $\Sigma_2^0(D)$ -complete.

The final statement is immediate from the definitions. \square

This gives a stark dichotomy for the Turing degrees. For example, if T is a non-standard completion of PA, every Turing degree either uniformly witnesses the Π_ω^0 -hardness of $\text{Mod}(T)$ or fails to even witness the 2^ω -hardness of $\text{Mod}(T)$.

We now turn to the degree of the oracle needed for Lemma 4.7, with the same result being true for Lemma 4.8:

Theorem 6.4. *Let T be a (possibly incomplete) theory without a \forall_n -axiomatization. Let D be PA over T . Then D uniformly witnesses the Σ_n^0 -hardness of $\text{Mod}(T)$.*

Proof. Computable from the degree \mathbf{d} , we fix an effective enumeration R of a Scott set \mathcal{S} containing T . Next, we note that there are a pair of theories T_0 and T_1 as in Lemma 4.7 inside \mathcal{S} : There is a T -computable tree of viable choices for $T_0 \oplus T_1$, and the existence of any such pair shows that the tree is infinite. Next we apply Lemma 4.1 to this pair T^- and T^+ . Note that in Section 5.2, the oracle used is exactly $C \oplus R \oplus p$, where R is an enumeration of a Scott set \mathcal{S} which contains both T^- and T^+ . Thus we can take Ψ to be the construction sending p to \mathcal{M}_p in the proof of Lemma 4.1 applied to the pair T^- and T^+ . \square

We note that again this is optimal since, with T being an incomplete theory, it may take a degree PA over T just to build any model of T .

Finally, we observe that \emptyset does not witness Σ_2^0 -hardness for completions of even very weak fragments of PA.

Definition 6.5. Let \exists_1^{\leq} be the set of formulas in the language of Peano arithmetic containing all bounded existential formulas. We let $I\exists_1^{\leq}$ be the set of induction principles for these formulas. For $\varphi \in \exists_1^{\leq}$, the typical axiom in $I\exists_1^{\leq}$ is of the form

$$(\forall x(\varphi(x) \rightarrow \varphi(x+1)) \wedge \varphi(0)) \rightarrow \forall x \varphi(x).$$

Theorem 6.6 (Wilmer [Wi85]). *If $\mathcal{M} \models I\exists_1^{\leq}$, then \mathcal{M} is computable if and only if $\mathcal{M} \cong \mathbb{N}$, the standard model of the natural numbers.*

Theorem 6.7. *Let T be a complete consistent extension of $\text{PA}^- + I\exists_1^{\leq}$. Then $\text{Mod}(T)$ is Π_ω^0 -complete, but \emptyset does not witness the Σ_2^0 -hardness of $\text{Mod}(T)$. If T is not TA, then \emptyset does not witness the 2^ω -hardness of $\text{Mod}(T)$.*

Proof. By the recent work of Enayat and Visser [EVta], no complete consistent extension of PA^- has a bounded axiomatization, so $\text{Mod}(T)$ is Π_ω^0 -complete by Theorem 3.2. Yet, by Theorem 6.6, \emptyset computes no non-standard model of T . Thus Corollary 6.3 shows that \emptyset does not witness the Σ_2^0 -hardness of $\text{Mod}(T)$, and if T is not TA, then \emptyset does not witness the 2^ω -hardness of $\text{Mod}(T)$. \square

Shepherdson gave examples [Sh64] of computable non-standard models of quantifier-free induction, which suggests that Theorem 6.7 might fail for completions of PA^- . We thus ask:

Question 6.8. Is there a completion T of PA^- such that \emptyset witnesses the Π_ω^0 -hardness of $\text{Mod}(T)$?

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