

ANALOGUES OF THE COUNTABLE BOREL EQUIVALENCE RELATIONS IN THE SETTING OF COMPUTABLE REDUCIBILITY

ABSTRACT. Coskey, Hamkins, and Miller [CHM12] proposed two possible analogues of the class of countable Borel equivalence relations in the setting of computable reducibility of equivalence relations on the computably enumerable (c.e.) sets. The first is based on effectivizing the Lusin/Novikov theorem while the latter is based on effectivizing the Feldman/Moore theorem. They asked for an analysis of which degrees under computable reducibility are attained under each of these notions.

We investigate these two notions, in particular showing that the latter notion has a strict dichotomy theorem: Every such equivalence relation is either equivalent to the relation of equality ($=^{ce}$) or almost equality (E_0^{ce}) between c.e. sets. For the former notion, we show that this is not true, but rather there are both chains and antichains of such equivalence relations on c.e. sets which are between $=^{ce}$ and E_0^{ce} . This gives several strong answers to [CHM12, Question 3.5] showing that in general there is no analogue of the Glimm-Effros dichotomy for equivalence relations on the c.e. sets.

1. INTRODUCTION

Invariant descriptive set theory [Gao08] studies the complexity of equivalence relations up to Borel reducibility. Such a theory serves as a theoretical framework for investigating the complexity of classification problems naturally arising in mathematics. A fundamental subclass of Borel equivalence relations is that of *countable Borel equivalence relations* (cbers), i.e., those whose equivalence classes are countable. By the Feldman/Moore theorem [FM77], it turns out that cbbers are exactly the orbit equivalence relations generated by Borel actions of countable groups and this brings into the subject deep connections with group theory, ergodic theory, and operator algebras (see, e.g., [JKL02, Kec19]). The Feldman/Moore theorem is a straightforward consequence of a classic uniformization result, due to Lusin and Novikov, which ensures that all cbbers have a uniform Borel enumeration of each class.

Paradigmatic examples of cbbers are the identity relation on a given standard Borel space X , denoted $\text{Id}(X)$, and the eventual equality on the Cantor space 2^ω , denoted E_0 . It follows from Silver's dichotomy [Sil80] that, if X is uncountable, then $\text{Id}(X)$ is Borel reducible to any cber on X . Moreover, the Glimm-Effros dichotomy [HKL90] states that E_0 is a successor of $\text{Id}(2^\omega)$ in the Borel hierarchy, i.e., every Borel equivalence relation is either reducible to $\text{Id}(2^\omega)$ or E_0 reduces to it. Beyond E_0 , the Borel hierarchy of cbbers is much wilder: e.g., Loveau and Velickovic [LV94] proved that it contains both infinite chains and antichains. Yet, there exists a universal cber E_∞ to which all cbbers reduce [DJK94].

Coskey, Hamkins, and Miller [CHM12] suggested to effectivize set theoretic Borel equivalence relations by restricting the focus to their computably enumerable (c.e.) instances. By identifying c.e. sets with their indices, this restriction allows to project equivalence relations from 2^ω to ω . So, e.g., $\text{Id}(2^\omega)$ and E_0^{ce} translate, respectively, to the equality of c.e. sets, denoted by $=^{ce}$, and to the almost equality of c.e. sets, denoted by E_0^{ce} .

In [CHM12], two effective analogues of the class of cbers are proposed. Roughly (formal definitions will be given below): the *c.e. orbit equivalence relations*, which are based on effectivizing the Feldman/Moore theorem, are those arising from a computable group acting, in a suitable way, on the c.e. sets; the *equivalence relations enumerable in the indices*, on the other hand, are based on effectivizing the Lusin/Novikov theorem.

Coskey, Hamkins, and Miller [CHM12] proved that, contrary to the Borel case, the two notions do not align. They showed that the equivalence relation E_0^{ce} is enumerable in the indices but no suitable action on the c.e. sets realizes it. They asked whether E_0^{ce} is computably bireducible with a c.e. orbit equivalence relation. More generally, they asked for a degree theoretic analysis of these notions under computable reducibility, the most popular tool for classifying equivalence relations on ω . In this paper, we offer such analysis. Our main theorem expresses a sharp and quite unexpected dichotomy:

Dichotomy Theorem. *Up to computable reducibility, every c.e. orbit equivalence relation is either equivalent to $=^{ce}$ or E_0^{ce} .*

Hence, c.e. orbit equivalence relations are much more well-behaved than their Borel counterpart. This is in sharp contrast with the evidence that many desirable properties of a poset fail for degree structures based on computable reducibility (such as **Ceers** [GG01, AS19, ASS20] and **ER** [ABSM21]). On the other hand, the property of being enumerable in the indices gives rise to a more complicated hierarchy: Theorems 4.1 and 4.3 state that, between $=^{ce}$ and E_0^{ce} , there are both infinite chains and infinite antichains of equivalence relations which are enumerable in the indices. It follows that there is no analogue of the Glimm-Effros dichotomy for equivalence relations on the c.e. sets, which gives a strong solution to [CHM12, Question 3.5].

1.1. Preliminaries. We assume that the reader is familiar with the fundamental notions and techniques of computability theory. In particular, we shall freely use the standard machinery for priority arguments, (e.g., strategies, requirements, outcomes, injury, tree of strategies), as is surveyed in [Soa87].

Group Actions. Let G be a group acting on some set X . Let $\pi : G \rightarrow S_X$ be the induced permutation representation. We say G has *only finitely many actions* if $\text{ran}(\pi)$ is finite. For any $Y \subseteq X$, we let $\text{Stab}(Y) = \{g \in G : \forall x \in Y (g \cdot x = x)\}$. We say G has *isolated actions* if there is a finite set $F \subseteq X$ so that $\text{Stab}(F) = \ker(\pi)$. The *orbit equivalence relation* E_G on X is given by $x E_G y \Leftrightarrow (\exists g \in G)(g \cdot x = y)$. Equivalence classes of E_G are called G -orbits. For a set $S \subseteq X$, we let $G \cdot S$ be $\{g \cdot x : x \in S\}$; if $G \cdot S$ consists of a single G -orbit, we say that G acts *transitively* on S . Similarly, if $S \subseteq X$ and $g \in G$, we let $g \cdot S = \{g \cdot x : x \in S\}$.

The next few easy group theoretic lemmas will facilitate our classification of the c.e. orbit equivalence relations.

Lemma 1.1. *If G has non-isolated actions, then for any $g \in G$ and F a finite subset of ω , there is $h \in G$ so that $h \upharpoonright F = g \upharpoonright F$ and $h \circ g^{-1} \notin \ker(\pi)$.*

Proof. Let $h_0 \in \text{Stab}(F) \setminus \ker(\pi)$ and let $h = h_0 \circ g$. □

Lemma 1.2. *If G has isolated actions and all G -orbits are finite, then G has only finitely many actions.*

Proof. Let F be finite so $\text{Stab}(F) = \ker(\pi)$. Any $f, g \in G$ that act the same on F have $f^{-1}g \in \ker(\pi)$, so $\pi(f) = \pi(g)$. But since each $x \in F$ has a finite G -orbit, there are only finitely many total possible images for F for any action in G . □

Lemma 1.3. *Suppose G acts on some infinite set S transitively. Then, for any finite set $F \subseteq S$, there is $g \in G$ so that $g \cdot F$ is disjoint from F .*

Proof. For each pair $x, y \in F$, the set of $g \in G$ so that $g \cdot x = y$ is a coset of $\text{Stab}(x)$. Then, since S is infinite and G acts transitively, the $\text{Stab}(x)$ must have infinite index. By Neumann's lemma [Neu54], a group cannot be covered by a finite union of cosets of subgroups of infinite index. \dagger \square

Computable Reducibility. For equivalence relations E and F on ω , E is *computably reducible* to F , written $E \leq_c F$, if there is a computable function f so that $x E y \Leftrightarrow f(x) F f(y)$. Henceforth, we refer to computable reductions as just reductions. We write $E \equiv_c F$, if E and F reduce to each other.

Definition 1.4 ([CHM12]). Let $(W_e)_{e \in \omega}$ be a uniform enumeration of all c.e. sets, and denote by **CE** the collection of c.e. subsets of ω .

- If E is an equivalence relation on 2^ω , then E^{ce} is an equivalence relation on ω given by

$$e E^{ce} i \Leftrightarrow W_e E W_i.$$

Note that every E^{ce} is a *quotient* of $=^{ce}$, i.e., $=^{ce} \subseteq E^{ce}$.

- An action of a computable group G on **CE** (note that the action is on the collection of sets, not on indices for these sets) is *computable in indices* if there is computable $\alpha : G \times \omega \rightarrow \omega$ so that

$$W_{\alpha(\gamma, e)} = \gamma \cdot W_e.$$

We use the term *c.e. orbit equivalence relation* and the notation E_G^{ce} to mean an orbit equivalence relation of a group action on **CE** which is computable in indices.

- E^{ce} is *enumerable in the indices* if there is a computable $\alpha : \omega \times \omega \rightarrow \omega$ so that

$$i E^{ce} j \Leftrightarrow (\exists n)(W_{\alpha(i, n)} = W_j).$$

It is easy to see that $=^{ce}$ reduces to E_0^{ce} . To see that E_0^{ce} does not reduce to $=^{ce}$, it suffices to observe that $=^{ce}$ is Π_2^0 while E_0^{ce} is strictly Σ_3^0 (see [CHM12, Theorem 3.4]). In fact, the following holds:

Theorem 1.5 (Ianovski, Miller, Ng, Nies [IMNN14]). *E_0^{ce} is a universal Σ_3^0 equivalence relation under computable reducibility.*

Note that both the c.e. orbit equivalence relations and the equivalence relations enumerable in the indices are subclasses of Σ_3^0 equivalence relations. Thus, they all reduce to E_0^{ce} . Finally, c.e. equivalence relations, widely investigated in literature (see, e.g., [GG01, AS19, ASS20, ASM22]), are called *ceers*.

In Section 2, we show that any action on **CE** which is computable in indices is induced by a computable permutation group acting on ω . Using this, in Section 3 we prove the dichotomy theorem that every c.e. orbit equivalence relation is equivalent to either $=^{ce}$ or E_0^{ce} . To do this, we show that every action comes in one of three types and we prove the result for each of these types in subsections 3.1-3.3. Finally, in Section 4 we consider the equivalence relations enumerable in the indices and show that there are infinite chains and antichains of these between $=^{ce}$ and E_0^{ce} .

\dagger We thank Meng-Che Ho for pointing out this slick proof.

2. REDUCING TO COMPUTABLE PERMUTATION GROUPS

We begin by proving a simple yet fundamental lemma that describes how the Recursion Theorem constrains the behavior of group actions which are computable in indices. From this lemma it will follow that, without loss of generality, we may assume that any c.e. orbit equivalence relation is naturally induced by a computable permutation group on ω (i.e., a computable subgroup of S_∞).

Notation. Throughout this section, we let α be a computable function witnessing that a given group G acts computably in indices on **CE**.

Lemma 2.1. *For each $\gamma \in G$ and c.e. sets U, V*

- (1) $U \subseteq V \Rightarrow \gamma \cdot U \subseteq \gamma \cdot V$;
- (2) $|V| = |\gamma \cdot V|$.

Proof. (1) Suppose that $U \subseteq V$ and take any $n \in \gamma \cdot U$. Let e be an index we control by the Recursion Theorem. We copy U into W_e , unless we see n enter in $W_{\alpha(\gamma, e)}$ in which case we copy V in W_e . We must have $n \in W_{\alpha(\gamma, e)}$ as otherwise $W_e = U$ and $n \in \gamma \cdot U = W_{\alpha(\gamma, e)}$. Thus $n \in W_{\alpha(\gamma, e)}$ and $W_e = V$. This shows $n \in W_{\alpha(\gamma, e)} = \gamma \cdot V$.

(2) Suppose that $|V| > |\gamma \cdot V|$. Since the action of γ must be injective on **CE** and (1) holds, there is simply not enough room to accommodate all subsets of V into the subsets of $\gamma \cdot V$. To exclude that $|V| < |\gamma \cdot V|$, just note that $V = \gamma^{-1} \cdot (\gamma \cdot V)$. \square

Definition 2.2. For each $\gamma \in G$, let the function $F_\gamma : \omega \rightarrow \omega$ be given by

$$F_\gamma(n) = m \Leftrightarrow \gamma \cdot \{n\} = \{m\}.$$

Lemma 2.3. *For all $\gamma \in G$, F_γ is a computable permutation of ω . Moreover, $\gamma \cdot V = \{F_\gamma(n) : n \in V\}$ for each c.e. V .*

Proof. We first observe that F_γ is a permutation of ω . Since γ acts injectively on **CE**, F_γ must be injective. But since F_γ is necessarily the inverse of $F_{\gamma^{-1}}$, we see it is also surjective. This permutation of ω is computable since we can just wait to see which number enters $W_{\alpha(\gamma, i)}$ for a chosen i so that $W_i = \{n\}$.

Next, by Lemma 2.1(1), $\gamma \cdot V \supseteq \{F_\gamma(n) : n \in V\}$. Applying the same to γ^{-1} , we see that $\gamma \cdot V = \{F_\gamma(n) : n \in V\}$. \square

This allows us consider the c.e. orbit equivalence relations in a more concrete fashion:

Definition 2.4. For G a computable subgroup of S_∞ , let

$$i R_G^{ce} j \Leftrightarrow (\exists \gamma \in G)(W_i = \{\gamma(x) : x \in W_j\}).$$

The next lemma, which follows directly from Lemma 2.3, ensures that focusing only to c.e. orbit equivalence relations of the form R_G^{ce} is not restrictive:

Lemma 2.5. *For every c.e. orbit equivalence relation E_G^{ce} , there is a computable subgroup H of S_∞ so that $E_G^{ce} = R_H^{ce}$.*

3. THE DICHOTOMY THEOREM FOR C.E. ORBIT EQUIVALENCE RELATIONS

This section is devoted to the proof of the dichotomy theorem: We show that every R_G^{ce} is either Σ_3^0 complete (and thus, by Theorem 1.5, equivalent to E_0^{ce}) or reduces to $=^{ce}$. First, we note that every R_G^{ce} lies above $=^{ce}$.

Theorem 3.1. *There is a reduction f of $=^{ce}$ to itself so that, if $W_i \neq W_j$, then $W_{f(i)}$ is not computably isomorphic to $W_{f(j)}$. In particular, we have that f reduces $=^{ce}$ to any R_G^{ce} .*

Proof. We will construct a sequence of sets $(V_k)_{k \in \omega}$ so that $W_i = W_j$ implies $V_i = V_j$ and $W_i \neq W_j$ implies V_i and V_j are not computably isomorphic. To do so, we shall satisfy the following requirements:

- $\mathcal{R}_{i,j}$: If $W_i = W_j$, make $V_i = V_j$;
- $\mathcal{D}_{i,j}^n$: Make $V_j \neq \varphi_n(V_i)$.

The \mathcal{R} -strategies have two outcomes: $\infty < f$. Similarly, $\mathcal{D}_{i,j}^n$ -strategies have outcomes: $d < w$. We place these outcomes on a tree of strategies \mathbf{T} meeting the following conditions: Every path contains an $\mathcal{R}_{i,j}$ -node α before any $\mathcal{D}_{i,j}^n$ strategy. Every path extending $\alpha \frown f$ contains strategies for $\mathcal{D}_{i,j}^n$, for each n . No $\mathcal{D}_{i,j}^n$ -strategy extends $\alpha \frown \infty$.

The strategy to meet \mathcal{R} -requirements. For $\mathcal{R}_{i,j}$ -strategies, we use the usual computable approximation to determine if $W_i = W_j$. When the length of agreement of $W_{i,s}$ and $W_{j,s}$ changes, we take outcome ∞ and act as follows: we replace $V_{i,s}$ by $V_{i,s} \cup (V_{j,s} \cap [0, \ell])$ and $V_{j,s}$ by $V_{j,s} \cup (V_{i,s} \cap [0, \ell])$, where ℓ denotes the length of agreement of $V_{i,s}$ and $V_{j,s}$. On all other stages, let x be the least element so $W_{i,s}(x) \neq W_{j,s}(x)$. We say this is an (i, j) -stage if $x \in W_{i,s} \setminus W_{j,s}$ and it is a (j, i) -stage if $x \in W_{j,s} \setminus W_{i,s}$.

The strategy to meet \mathcal{D} -requirements. A $\mathcal{D}_{i,j}^n$ -strategy α acts as follows: Assume that this is an (i, j) -stage (if it is a (j, i) -stage instead, reverse the role of i and j using φ_n^{-1} instead of φ_n). Also, suppose there are M numbers restrained by higher priority \mathcal{D} -strategies.

First, we choose $M + 1$ new numbers K_m^α for $m < M + 1$ and restrain K_m^α from entering any set V_k . We wait for $\varphi_n(K_m^\alpha)$ to converge for each m and take outcome w . Once $\varphi_n(K_m^\alpha)$ converges for each m , we take outcome d and act as follows: If there are $m_0 < m_1 < M + 1$ so that $\varphi_n(K_{m_0}^\alpha) = \varphi_n(K_{m_1}^\alpha)$, then we do nothing since φ_n is not a computable permutation. Otherwise, we choose one number $\varphi_n(K^\alpha)$ which is not restrained by a higher priority \mathcal{D} -strategy. We act depending on the value of $\varphi_n(K^\alpha)$:

- (a) If $\varphi_n(K^\alpha) = K_e^\beta$ for a lower priority β (i.e., a strategy β being injured by our taking outcome d) or if $\varphi_n(K^\alpha)$ is not chosen as K_e^β for any β , then we place $\varphi_n(K^\alpha)$ into V_j .
- (b) If $\varphi_n(K^\alpha) = K^\alpha$, then we place K^α into V_i (this is the case where it being an (i, j) -stage matters).

The verification: The verification is based on the following lemmas.

Lemma 3.2. *If α places a restraint against K_m^α entering any set, and α has not acted or been injured, then K_m^α has not entered any set. If α acts under case (a) and is never injured, then K_m^α still never enters any set.*

Proof. Since K_m^α is not in any V_i , it cannot enter any V_i via action for an \mathcal{R} -strategy. Only a higher priority \mathcal{D} -strategy or α itself would put K_m^α into any set. In the former case, α would be injured by this, and in the latter case, α acts. If α acts under case (a), then α also does not put K_m^α into any V_i . \square

Lemma 3.3. *If α puts K_m^α into V_i by case (b), then either α is injured or K_m^α never enters V_j (symmetrical if the stage is a (j, i) -stage).*

Proof. For K_m^α to enter V_j , there must be some sequence of sets V_{k_0}, \dots, V_{k_n} so that $V_i = V_{k_0}$, $V_j = V_{k_n}$ and at some stage after stage s , we must approximate that $W_{k_m} = W_{k_{m+1}}$ with lengths of agreement at least K_m^α . But consider the length of agreement ℓ of $W_{i,s}$ and $W_{j,s}$. We had $\ell \in W_{i,s} \setminus W_{j,s}$ since it was an (i, j) -stage. If the length of agreement changed, then α would have been injured. But then we must see ℓ enter each successive $W_{k_{m+1}}$ until we see ℓ enter $W_{k_n} = W_j$, so the length of agreement would change after all. \square

Lemma 3.4. *If α is on the true path, then α ensures that its requirement is satisfied.*

Proof. If α is an \mathcal{R} -requirement, it need only succeed if the true outcome is infinite. In this case, as the length of agreement between $W_{i,s}$ and $W_{j,s}$ goes to infinity, we ensure that more and more of V_i and V_j agree on a cofinal set of stages, ensuring that $V_i = V_j$.

Next, suppose that α is a $\mathcal{D}_{i,j}^n$ -requirement. We consider a stage s late enough that α is never injured after s and α acts at stage s if it ever will. If the true outcome is w , then φ_n is not a permutation and the requirement is satisfied. If the true outcome is d , we must consider the two cases above: In case (a), Lemma 3.2 shows that $\varphi(K_m^\alpha) \in V_j \setminus V_i$ and in case (b) Lemma 3.3 shows that $\varphi(K_m^\alpha) \in V_i \setminus V_j$ (symmetrical if it is a (j, i) -stage). \square

This concludes the proof that f reduces $=^{ce}$ to any R_G^{ce} . \square

We just proved that $=^{ce}$ is the least c.e. orbit equivalence relation. To calculate the complexity of all R_G^{ce} 's and obtain the dichotomy result, we shall now separate three cases: (i) G has only finitely many actions; (ii) There is an infinite G -orbit; (iii) The actions in G are not isolated. Lemma 1.2 guarantees that there are no other cases to be considered.

3.1. Case (i): G has only finitely many actions. Since all equivalence relations R_G^{ce} 's arise from permutation groups, we may assume that G is finite. To facilitate our analysis, we introduce the following equivalence relation which turns out to be equivalent to $=^{ce}$.

Definition 3.5. Let E_{set}^n be given by $i E_{set}^n j$ if and only if $i = \langle i_0, \dots, i_{n-1} \rangle$ and $j = \langle j_0, \dots, j_{n-1} \rangle$ and $\{W_{i_k} : k < n\} = \{W_{j_k} : k < n\}$ where $\langle \cdot, \dots, \cdot \rangle$ is an n -ary pairing function.

Lemma 3.6. *If G has only finitely many actions, then R_G^{ce} reduces to E_{set}^n for some n .*

Proof. Let g_0, \dots, g_{n-1} be group elements representing all distinct actions in G . We obtain a reduction of E_G to E_{set}^n by the map which sends any c.e. set W_i to an index for the family $\{g_0 \cdot W_i, \dots, g_{n-1} \cdot W_i\}$. \square

We now need to show that E_{set}^n reduces to $=^{ce}$.

Theorem 3.7. *For each n , E_{set}^n reduces to $=^{ce}$.*

Proof. Let h be a function which sends $i = \langle i_0, \dots, i_{n-1} \rangle$ to a c.e. index for the set

$$V_i = \{(k, \rho_0, \dots, \rho_{n-1}) : \text{each } \rho_i \in 2^k \text{ and } (\exists \pi \in S_n)(\forall i < k)(\rho_i \subseteq W_{\sigma(i)})\}.$$

It is immediate that if $i E_{set}^n j$ then $V_i = V_j$. On the other hand, if $V_i = V_j$, then for every k there is some $\pi_k \in S_n$ so that $(\forall l < n)(W_{i_l} \upharpoonright k = W_{j_{\pi_k(l)}} \upharpoonright k)$. Hence, by the pigeonhole principle, there is some permutation $\pi \in S_n$ so that $(\forall l < n)(W_{i_l} = W_{j_{\pi(l)}})$. \square

Putting together Lemma 3.6, Theorem 3.7, and Theorem 3.1, we get that c.e. orbit equivalence relations induced by finite permutation groups are as simple as possible:

Theorem 3.8. *If G contains only finitely many actions, then $R_G^{ce} \equiv_c =^{ce}$.*

3.2. Case (ii): There is an infinite G -orbit.

Theorem 3.9. *If there is an element $n \in \omega$ so that the G -orbit of n is infinite, then R_G^{ce} is Σ_3^0 complete.*

Proof. We will do our coding within the orbit of n , which forms an infinite c.e. set. Using a computable bijection between this set and ω , we assume with no loss of generality that G acts transitively on ω .

We fix R a Σ_3^0 equivalence relation defined by $i R j$ if and only if $\exists n X(i, j, n)$ for a Π_2^0 relation X . We construct a uniformly c.e. sequence of sets $V_i = W_{f(i)}$ for $i \in \omega$ so that $i R j$ if and only if $f(i) R_G^{ce} f(j)$. That is, there is an n so that $X(i, j, n)$ if and only if there is some $g \in G$ so that $V_i = g(V_j)$.

The requirements and their interaction. For every $i < j \in \omega$ and $n \in \omega$, we have the following requirements:

$\mathcal{P}_{i,j}^n$: If $X(i, j, n)$, then make $V_j = g(V_i)$ for some $g \in G$. If $\neg X(i, j, n)$ then ensure that either φ_n is not a permutation of ω or $V_j \neq \varphi_n(V_i)$.

Each strategy has three outcomes: $\infty < d < w$. Outcome ∞ represents $X(i, j, n)$, outcome d represents $\neg X(i, j, n)$ and we succeed in diagonalizing to ensure $V_j \neq \varphi_n(V_i)$, and outcome w represents $\neg X(i, j, n)$ and we never get a chance to diagonalize since we are waiting for φ_n to converge.

We put these strategies on a tree so that, if τ is given the requirement $\mathcal{P}_{i,j}^n$, then we place no strategy $\mathcal{P}_{i',j'}^k$ or $\mathcal{P}_{j,i}^k$ below $\tau \frown \infty$. We do this so that for every $\sigma \in \{\infty, d, w\}^\omega$ and $j \in \omega$, either there is precisely one τ so that $\tau \frown \infty \prec \sigma$ and τ is a $\mathcal{P}_{i,j}^k$ -strategy for some $i < j$ and k , or for every $i < j$ and k , there is some τ so that τ is a $\mathcal{P}_{i,j}^k$ -strategy and either $\tau \frown d \prec \sigma$ or $\tau \frown w \prec \sigma$.

When first visited, a $\mathcal{P}_{i,j}^n$ -strategy will choose an element $g \in G$ which it will use in its infinite outcome. Its choice of g must be consistent with the rest of the construction. In particular, if it applies the infinite outcome, it does not want to cause injury to any higher priority requirement's diagonalization. Namely, for the purpose of diagonalizing, a strategy $\mathcal{P}_{i',j'}^{n'}$ will choose a number K and it may put K into $V_{i'}$ and attempt to keep K out of $V_{j'}$. On the other hand, under the infinite outcome, for the purpose of ensuring $V_j = g(V_i)$, $\mathcal{P}_{i,j}^n$ will act by putting $g^{-1}(V_j)$ into V_i and putting $g(V_i)$ into V_j . We must ensure that the cumulative effect of these infinite outcomes will not ruin the $\mathcal{P}_{i',j'}^{n'}$ -strategy diagonalization. That is, that they will not put $\varphi_{n'}(K)$ into $V_{j'}$.

To ensure this, we will define the set F of numbers currently relevant to the construction. That includes all those K 's which might enter some set for the sake of a diagonalization and also all those numbers N so that N entering any V_ℓ might possibly cause $\varphi_n(K)$ to enter $V_{j'}$ by the actions of all the currently active strategies. Then, we will rely on Lemma 1.3 to choose an element $g \in G$ so that $g \cdot F \cap F = \emptyset$.

Definition 3.10. At all stages s of the construction, for any given node α , we define the set of α -restrained pairs as follows: If α restrains a number n from entering a set V_j , then the pair (n, j) is a restrained pair. In addition, we say a pair (m, k) is α -restrained if there is a sequence of currently active nodes on the tree β_0, \dots, β_n such that each β_i is either so that

- For each i , either $\beta_i \frown \infty \preceq \alpha$ or $\alpha <_L \beta$ or $\alpha \frown d \preceq \beta$ or $\alpha \frown w \preceq \beta$.
- If $i < j$, then β_i taking outcome ∞ does not injure β_j .
- If m were in V_k and then each of these β_i were to take their infinite outcomes, in order, it would cause n to enter V_j .

The strategy to meet $\mathcal{P}_{i,j}^n$ at node τ . When initialized, let F be the set of numbers mentioned so far in the construction, including every n which is in an α -restrained pair for any α and every requirement's parameter K or $\varphi_n(K)$ (if already defined). We will show in Lemma 3.12 that this is a finite set. Applying Lemma 1.3, choose a g in G so that $g \cdot F \cap F = \emptyset$.

- When we approximate that $X(i, j, n)$ holds, this strategy will attempt to make $V_j = g(V_i)$. In this case, we take the outcome ∞ and the node acts by putting $g^{-1}(V_j^s)$ into V_i and $g(V_i^s)$ into V_j ;
- When we approximate that $\neg X(i, j, n)$ holds, we employ the following strategy towards ensuring $V_j \neq \varphi_n(V_i)$. We first choose a new number K , in particular K is not in any α -restrained pair for any currently active α . Moreover, after placing this restraint, there should be no τ -restrained pair (m, l) with $m \in V_l$. We prove in Lemma 3.13 that such a K can be chosen. We place the restraint that K should not enter V_i , and wait for $\varphi_n(K)$ to converge. While we wait, we take outcome w .
Once we see $\varphi_n(K)$ converge, we check if we can place a restraint keeping $\varphi_n(K)$ from entering V_j . That is, we check if, once we place this restraint we would have an α -restrained pair (m, l) with m already in V_l . If so, we do nothing. If not, we place K into V_i and we place the restraint that $\varphi_n(K)$ should not enter V_j . In this case, we take outcome d .

Verification: The verification is based on the following lemmas.

Lemma 3.11. *If α is a $\mathcal{P}_{i,j}^n$ -strategy on the true path with true outcome ∞ , then $i E_G j$.*

Proof. Let s be a stage when α is last initialized. Then α chooses a group element g . Infinitely often, when α is visited, it puts $g^{-1}(V_j^s)$ into V_i and $g(V_i^s)$ into V_j . This ensures that $V_j = g(V_i)$ and thus $i E_G j$. \square

Lemma 3.12. *At each stage s , there are only finitely many α -restrained pairs.*

Proof. We note that if $\beta \succeq \gamma \frown \infty$, then we do not have γ as a $\mathcal{P}_{i,j}^k$ -strategy and β as a $\mathcal{P}_{i',j}^l$ -strategy or $\mathcal{P}_{j,m}^l$ -strategy. Thus, if we put an element into V_j for some j , and via some sequence of infinite outcomes, that causes some other element to appear in V_j , this must have been due to some injury among the requirements. As there are only finitely many currently active requirements, this can happen only finitely often. Thus, among the finitely many sets V_i currently under consideration, the set of pairs (m, j) which might cause α 's restrained number n to enter the set V_j that it is restrained from, is a finite set. \square

Lemma 3.13. *At any stage s , active node α , and $i \in \omega$, there are only finitely many K so that α placing a restraint against K entering V_i would cause there to be an α -restrained pair (x, m) with x already in V_m .*

Proof. As in Lemma 3.12, each x entering V_m can cause at most finitely many numbers to enter V_i via a sequence of infinite outcomes of currently active nodes. As there are only finitely many numbers at stage s already in $\bigcup_{m \in \omega} V_m$, this makes only finitely many K have the property that a restraint against K entering V_i would cause there to be an α -restrained pair (x, m) with x already in V_m . \square

Lemma 3.14. *At every stage of the construction, if α is an active node restraining a from entering to V_b , then there is no α -restrained pair (m, l) so that m is in V_l .*

Also, there is no α -restrained pair (m, l) so that $m = K^\beta$ for some $\beta \frown d \not\prec_L \alpha \frown d$ which is an $\mathcal{P}_{i,l}^c$ -strategy.

Proof. We prove both claims by simultaneous induction. We begin our induction at the moment when α places its restraint. At this point, no α -restrained pair (m, k) can have m in V_k . If the restrained pair is (K, i) , this is true because K is chosen to be new. If the restrained pair is $(\varphi_n(K), j)$, we ensure this condition before placing the restraint. The second condition is ensured at this moment because when α places its restraint, it takes outcome w or d and thus the only active nodes β which currently have a parameter K^β must have $\beta \sim d$ to the left of $\alpha \sim d$.

As moments of the construction go by, we have to check that we preserve the inductive hypotheses. We have four types of actions to consider:

- (1) New strategies choosing $g \in G$.
- (2) Other strategies taking the infinite outcome.
- (3) Other strategies taking their outcome d .
- (4) Other strategies choose a new K .

When a new strategy chooses its $g \in G$, it does so in a way to ensure that it maintains this inductive hypotheses. In particular if β is choosing its parameter g , then this is the first time β is visited since (re)initialization. Thus β is the rightmost active node. Thus to violate the inductive hypothesis, g would have to move either an element in one of the V_j 's or some K^γ to some element which is α -restrained. But g is chosen not to do this.

When other strategies take the infinite outcome, either this outcome injures α or this one step was already considered as a possible step before this happened. In particular, if this puts m into V_i , then it is because m' was already in V_i . Then (m, l) cannot be an α -restrained pair, because then (m', l') would have been an α -restrained pair contradicting the inductive hypothesis. Thus the first condition is preserved. As there can now be only fewer α -restrained pairs and no new parameters K^β have been chosen, the second condition is maintained as well.

When other strategies take their outcome d , they may put their number K into their set V_i . By the second condition of the inductive hypothesis, we have preserved the first condition of the induction hypotheses. As there can now be only fewer α -restrained pairs and no new parameters K^β have been chosen, the second condition is maintained as well.

When other strategies choose a new K , they do so in order to maintain these inductive hypotheses. In particular, K is chosen to not be in any α -restrained pair. This preserves the second condition, and the first condition is unchanged. \square

Lemma 3.15. *If α is a $\mathcal{P}_{i,j}^n$ -strategy on the true path with true outcome d or w , then $V_j \neq \varphi_n(V_i)$.*

Proof. If the true outcome is w , then φ_n is not total, so we consider the case where the true outcome is d . In this case, there are two possibilities to consider.

In the first possibility, we keep the restraint K shall not enter V_i . This is because we see, when visiting $\alpha \sim d$, that if we were to restrain $\varphi_n(K)$ from entering V_j we would already have some α -restrained pair (m, l) with m in V_i . Since all strategies right of α are reinitialized, as are all strategies below $\alpha \sim w$ or $\alpha \sim d$, this means that the strategies whose infinite outcomes are needed to move $m \in V_i$ to $\varphi_n(K) \in V_j$ are just those β so that $\beta \sim \infty \preceq \alpha$. Since we assume α is on the true path, the outcome $\beta \sim \infty$ will occur infinitely often, eventually we will see $\varphi_n(K) \in V_j$. By the first part of Lemma 3.14, we will never see K enter V_i , so we have successfully diagonalized.

In the second possibility, we put K into V_i , and we place restraint $\varphi_n(K)$ should not enter V_j . By the first part of Lemma 3.14, $\varphi_n(K)$ never enters V_j and we have successfully diagonalized. \square

Lemma 3.16. *For every $i < j$, $i R j$ if and only if $f(i) R_G^{ce} f(j)$.*

Proof. We claim by induction that $i R j$ if and only if $f(i) R_G^{ce} f(j)$. We assume the condition for all pairs (i', j') with $i', j' < j$ and consider pairs (i, j) with $i < j$. Suppose there is some $i < j$ with $i R j$. Then there must be some $i' < j$ so that a $\mathcal{P}_{i',j}^n$ -strategy τ is on the true path with true outcome ∞ . Then $i' < j$ and $i R i'$. By Lemma 3.11, the τ -strategy ensures $f(i') R_G^{ce} f(j)$. By inductive hypothesis, we have $f(i) R_G^{ce} f(i') R_G^{ce} f(j)$. Suppose that there is no $i < j$ with $i R j$. Then every $\mathcal{P}_{i,j}^n$ -strategy with $i < j$ has true outcome w or d . Thus Lemma 3.15 ensures that these strategies along the true path ensure that φ_n is not a bijection between $W_{f(i)}$ and $W_{f(j)}$. Together, these ensure that $f(i) R_G^{ce} f(j)$ for each $i < j$. \square

This concludes the proof that, if there is an infinite G -orbit, then R_G^{ce} is Σ_3^0 complete. \square

3.3. Case (iii): G has non-isolated actions. We first give a small reduction of our group G which maintains the properties we assume in this case and makes it computable to find indices for G -orbits.

Lemma 3.17. *If G is a computable group of permutations of ω so that 1_G is not isolated and all G -orbits are finite, then there is a computable subgroup G' of G so that $1_{G'}$ is not isolated and there is a computable function f sending $a \in \omega$ to a canonical index for the G -orbit of a .*

Proof. We construct G' as a c.e. subset of G as follows. We act in stages to satisfy requirements:

- $1_{G'}$ is not isolated.
- The G' -orbit of s will not grow after stage s .

To satisfy requirements of the second kind, we enumerate the orbit of the first s numbers a at stage s . That is, we take the set G_s of elements of G' that we have enumerated by stage s . Then, we compute the orbit of a in the finitely generated group generated by G_s . Call this set \mathcal{O}_a . We then place a restriction that we will, in the future, only consider elements g in G which have the property that $\forall x \in \mathcal{O}_a g(x) \in \mathcal{O}_a$. We note that a finite intersection of such subgroups forms an open subgroup H_S of G and so has the same properties that 1_{H_S} is not isolated and all orbits are finite.

A requirement of the first kind states that $G' \cap \text{Stab}(\{0, \dots, n\})$ has at least two elements. We find some element other than 1_G of the restricted group H_s (restricted due to requirements of the second kind) which is in $\text{Stab}(\{0, \dots, n\})$. We enumerate this element into G' . By proceeding as such, we enumerate a subset of G and we let G' be the subgroup generated by these. \square

Now we are ready to handle the case where G has non-isolated actions:

Theorem 3.18. *If G has non-isolated actions then R_G^{ce} is Σ_3^0 complete.*

Proof. We may assume that each G -orbit is finite, as otherwise the result follows from Theorem 3.9. We fix R a Σ_3^0 equivalence relation given by $\exists n X(i, j, n)$. Applying Lemma 3.17, we have a computable subgroup G' of G with uniformly computable orbits. In the proof below, we will construct a sequence of sets $V_i = W_{f(i)}$ and ensure that if $i R j$ then $f(i) E_G^{ce} f(j)$ and if $i \not R j$ then $V_i \neq \varphi_n(V_j)$ for every n . Thus we may replace G by G' and simply assume that G has uniformly computable finite orbits.

We again have strategies $\mathcal{P}_{i,j}^n$ placed on a tree with outcomes $\{\infty < d < w\}$.

$\mathcal{P}_{i,j}^n$: If $X(i, j, n)$, then make $V_j = g(V_i)$ for some $g \in G$. If $\neg X(i, j, n)$, then ensure that $V_j \neq \varphi_n(V_i)$.

A $\mathcal{P}_{i,j}^n$ -strategy α , after taking the d -outcome, will specify a computable set S_α of quadruples (a, b, k, ℓ) . This will be formed by taking the union of the S_β for higher priority strategies which have last taken their d -outcomes and possibly adding some new quadruples. A quadruple (a, b, k, ℓ) in a set S_α is understood as saying that the possible actions sending V_k to V_ℓ must send a to b . A potential map g from V_x to V_y is said to be *consistent with S_α* if for all quadruples $(a, b, x, y) \in S_\alpha$, we have $g(a) = b$.

A $\mathcal{P}_{i,j}^n$ -strategy will take outcome w when first visited. Whenever the approximation to X says that $X(i, j, n)$ holds, it will take outcome ∞ . Otherwise, it takes either outcome w or d , depending on whether a certain computation converges.

The strategy to meet a $\mathcal{P}_{i,j}^n$ -strategy at node τ . When first visited, τ sets S_τ to be the union of S_β for higher priority β . Then it chooses some $g^\tau \in G$ so that g is a potential map from V_i to V_j which is consistent with this S_τ . The strategy at τ then chooses a new number K and a pair of group elements g_0, g_1 each potential maps from V_i to V_j which are consistent with S_τ so that $g_0(K) \neq g_1(K)$. In choosing K to be new, we mean that K is chosen to be in a different G -orbit than any number previously mentioned in the construction. Then τ restrains any element from the orbit of K from entering any set V_ℓ . The strategy τ will continue to take outcome w until either the approximation to X says that $X(i, j, n)$, in which case it takes outcome ∞ or we see $\varphi_n(K) \downarrow$, in which case it will take outcome d . As long as it takes outcome w , it takes no further action.

When taking the outcome d for the first time since last taking outcome w , τ checks if $\varphi_n(K)$ is in $V_{j,s}$. If so, it does nothing and maintains its restraint against K entering V_i . If, on the other hand $\varphi_n(K)$ is not in $V_{j,s}$, then it puts K into V_i and restrains $\varphi_n(K)$ from entering V_j . For every single set V_ℓ , τ then puts exactly one member of the G -orbit of K into V_ℓ . This is done inductively as follows:

- (1) We put $g_0(K)$ into V_j unless $\varphi_n(K) = g_0(K)$, in which case we put $g_1(K)$ into V_j .
- (2) For each k , if there is a β so $\beta \frown \infty \preceq \tau$ and β is an $\mathcal{P}_{i',k}^m$ -requirement, then the number which we put in V_k is the g^β -image of the number that we put into $V_{i'}$.
- (3) For each k , if there is no β as such, then we choose any $h_k \in G$ a potential map from V_i to V_k which is consistent with S_τ and we put $h_k(K)$ into V_k .

Finally, we increase S_τ so that for every pair (k, ℓ) , we put (a, b, k, ℓ) into S_τ where a is the number we have put into V_k and b is the number we put into V_ℓ .

When taking the infinite outcome or if τ is injured, τ immediately places the entirety of the G -orbit of K into every set V_ℓ . If it has no parameter K chosen yet, then there is no clean-up to do here, and it does nothing. It also reverts its S_τ to being the union of the S_β for higher priority β . Note that τ does *not* perform any action on taking outcome ∞ to ensure that $g(V_i) = V_j$. In lieu of this, every strategy right of τ cleans up after themselves whenever they are injured, and strategies under the outcome $\tau \frown \infty$ respect τ 's choice of g^τ when they put numbers into V_j .

Verification: The verification is based on the following lemmas.

Lemma 3.19. *The choice of S_α is consistent and coherent. That is, for every pair k, ℓ , there is an element $g \in G$ which is a potential map from V_k to V_ℓ which consistent with S_α . Similarly, if $\delta \frown \infty \preceq \alpha$, then g^δ is consistent with S_α .*

Proof. We prove this by induction on stages.

As every S_β contains all the S_γ for γ higher priority, when S_α is first defined, it is set to equal some S_β already defined, so it is consistent by inductive hypothesis. Similarly, if $\delta \smallfrown \infty \preceq \alpha$, then also $\delta \smallfrown \infty \preceq \beta$ or $\beta = \delta$, thus the second condition is also maintained when S_α is first defined. Thus, we only need to check that when S_α grows due to taking action in the d outcome, the inductive hypotheses are maintained.

Let α be a $\mathcal{P}_{i,j}^n$ -strategy. It is immediate that coherence is maintained for the pair i, j , witnessed by either g_0 or g_1 . It is immediate that coherence is maintained for all pairs i, k where k is in case (3), since we chose an element $h_k \in G$ which was consistent with S_α to decide which element of the G -orbit of K to put into V_k . For all k in case (2), the second inductive hypothesis shows coherence for the pair i, k . Namely, we have two maps g^β and h_k each consistent with S_α , and their composition is also consistent with S_α . Finally, composing two maps between V_i and V_k and V_i and V_ℓ , we see that S_α is coherent for every pair k, ℓ .

Finally, we chose our elements to enter S_α so as to be consistent with g^β for all β with $\beta \smallfrown \infty \preceq \alpha$ in the second bullet. \square

This implies that when a node α is first visited, it can choose its parameter g^α .

Lemma 3.20. *If a node α places a restraint against a number n entering the set V_i , then either α is injured, lifts the restraint, or n does not enter V_i .*

Proof. When α places a restraint, it is either in outcome w or d . Note that the restraint is lifted if it ever enters outcome ∞ . So, the only strategies which can act, supposing that α is not injured and the restraint is not lifted are those to the right of $\alpha \smallfrown \infty$, which are all currently reinitialized, or nodes β so that $\beta \smallfrown \infty \preceq \alpha$. In any case, numbers only enter sets due to clean-up or diagonalization for elements K chosen after this restraint is placed. In particular, those elements are disjoint from the G -orbit of the restrained element, so in neither the diagonalization nor the clean-up can they cause the restrained number to enter any set. \square

Lemma 3.21. *Suppose τ is on the true path and is a $\mathcal{P}_{i,j}^n$ -strategy. If $X(i, j, n)$, then there is an element $g \in G$ so that $g(V_i) = V_j$. If $\neg X(i, j, n)$ then $\varphi_n(V_i) \neq V_j$.*

Proof. We first consider the case where $X(i, j, n)$ holds.

Consider the first stage s_0 at which τ is visited after its last initialization. Then it chooses an element $g^\tau \in G$. By choice of g^τ as being consistent with S_τ , it is consistent with all elements which have already entered V_i and V_j . That is, if a higher priority strategy β has placed x into V_i and y into V_j in the same orbit, then it put the quadruple (x, y, i, j) into its set S_β , and so we have $g^\tau(x) = y$.

No node to the left of τ ever acts again. The cumulative future effect of nodes right of $\tau \smallfrown \infty$ or above $\tau \smallfrown \infty$ are that they place entire G -orbits into V_i and V_j . Finally, we have to consider which elements might enter V_i and V_j due to strategies below $\tau \smallfrown \infty$. These place some number x into V_i , then, via the second bullet, they place $g^\tau(x)$ into V_j . Thus they also agree with g^τ . Thus $V_j = g^\tau(V_i)$.

Next, we consider the case where $\neg X(i, j, n)$. Consider the stage when τ takes outcome w after its last initialization and after its last time taking outcome ∞ . Then it chooses a number K and places restraint that no element in the G -orbit of K enter any set. If $\varphi_n(K)$ diverges, then the requirement is satisfied, so we may assume it converges. There are two cases, and Lemma 3.20 shows that $\varphi_n(K) \in V_j$ if and only if $K \notin V_i$ in either case. \square

As above, let f be so that $V_i = W_{f(i)}$ for all i . Then, the fact that f is a reduction follows by induction as in Lemma 3.16. \square

3.4. Concluding. Recall that, by Lemma 1.2, the three cases considered are exhaustive and, by Lemma 2.5, each c.e. orbit equivalent relation is of the form R_G^{ce} . Hence, putting Theorems 3.1, 3.9, and 3.18 together, we finally obtain the desired dichotomy:

Dichotomy Theorem. *Up to computable reducibility, every c.e. orbit equivalence relation is either equivalent to $=^{ce}$ or E_0^{ce} .*

4. EQUIVALENCE RELATIONS ENUMERABLE IN THE INDICES

We now give examples of degrees which contain equivalence relations which are enumerable in the indices. We concentrate on the interval between $=^{ce}$ and E_0^{ce} .

Contrasting with our dichotomy theorem for the case of a c.e. orbit equivalence relation, we show that there are both infinite chains and infinite antichains of equivalence relations which are enumerable in the indices. This gives several strong answers to [CHM12, Question 3.5] showing that there is no analogue of the Glimm-Efros dichotomy for equivalence relations on the c.e. sets.

Theorem 4.1. *There is an infinite chain of equivalence relations which are enumerable in the indices between $=^{ce}$ and E_0^{ce} .*

Proof. For any number i , let $F(i)$ be the least number in $\omega \setminus W_i$ and undefined if $W_i = \omega$. For each ceer X , let R_X be the equivalence relation defined as follows:

$$i R_X j \Leftrightarrow W_i = W_j \text{ or } (0 \in W_i \cap W_j \text{ and } F(i) X F(j)).$$

Lemma 4.2. *Let X be a ceer where every class is infinite. Then the relation R_X is enumerable in the indices.*

Proof. Until we see 0 enter W_i , we only enumerate i into the collection of indices equivalent to i . Once we see 0 enter W_i , we take our approximation y to $F(i)$ and for every $x \in [y]_X$, we enumerate all c.e. sets $(Y \cup [0, x - 1]) \setminus \{x\}$ into the family. If at a later stage we see y enter W_i , we take our new approximation y' to $F(i)$ and for every old set we choose an $x' > x$ so $x' X y'$ and x' hasn't already been enumerated into the set and we move to enumerating $(Y \cup [0, x' - 1]) \setminus \{x'\}$. In addition, we add a new column for each set $(Y \cup [0, z - 1]) \setminus \{z\}$ for any c.e. Y and $z X y'$. If we change our approximation to $F(x)$ infinitely often, each sets in our enumerated family is ω , which is exactly W_i . Otherwise, each set settles on something equivalent to W_i . Similarly, the last time we add columns, we add representatives of every c.e. set equivalent to W_i . \square

We apply the above lemma to the case of the ceer Id_n given by equivalence modulo n . For $n \geq 1$, let R_n be the equivalence relation given as R_{Id_n} . It follows that each R_n is enumerable in the indices. The map $k \mapsto l$ where $W_l = \{x + 1 : x \in W_k\}$ reduces $=^{ce}$ to R_n for each n as these sets do not contain 0. It suffices to show that $R_n <_c R_{n+1}$ for each n . The map $k \mapsto l$ where $W_l = \{(n + 1)x + y : nx + y \in W_k \text{ with } 0 \leq y < x\}$ gives a reduction of R_n to R_{n+1} . Finally, observe that R_{n+1} has precisely $n + 1$ classes which are properly Σ_2^0 . Namely, these are the classes of sets which contain 0 but are not total. But R_n only contains n such classes, so it follows that $R_{n+1} \not<_c R_n$. \square

Theorem 4.3. *There is an infinite antichain of equivalence relations enumerable in the indices between $=^{ce}$ and E_0^{ce} .*

Proof. We define the function $F(i)$ to be the least number k so that $2k + 1$ is not in W_i . We will construct a sequence $(X_n)_{n \in \omega}$ of ceers with infinite classes. Then we define a sequence A_n of equivalence relations as follows:

$$i A_n j \Leftrightarrow W_i = W_j \text{ or (both } |W_i \cap \text{Evens}| \text{ and } |W_j \cap \text{Evens}| \text{ are } \geq 2) \\ \text{or (both } W_i \text{ and } W_j \text{ contained only 1 even number } 2k \text{ and } \langle F(i), k \rangle X_n \langle F(j), k \rangle).$$

As in Lemma 4.2, each A_n is enumerable in the indices. In particular, whenever we believe W_i contains no even numbers, we only enumerate the index i as being equivalent to i . When we believe that W_i contains exactly 1 even number, we use the enumeration from Lemma 4.2, and as soon as we see that W_i contains at least two even numbers, we just enumerate indices for all c.e. sets containing at least two even numbers.

We note that there are four kinds of indices to consider:

- *Oddish indices:* These are i so that $W_i \subseteq \text{Odds}$. In this case, $[i]_{A_n} = [i]_{=ce}$, so $[i]_{A_n}$ is properly Π_2^0 . Using these indices and a function $k \mapsto \ell$ where $W_\ell = \{2x + 1 : x \in W_k\}$ is a reduction of $=^{ce}$ to each A_n .
- *Proper k -Coding indices:* These are i so that W_i contains only one even number $2k$ and $\text{Odds} \not\subseteq W_i$. Then $[i]_{A_n} = \{j : j \text{ is also a } k\text{-coding index and } \langle F(i), k \rangle X_n \langle F(j), k \rangle\}$. In this case $[i]_{A_n}$ is properly Σ_2^0 .
- *Full k -Coding indices:* These are i so that W_i contains only one even number $2k$ and $\text{Odds} \subseteq W_i$. In this case, $[i]_{A_n}$ is properly Π_2^0 .
- *Big indices:* These are i so that W_i contains ≥ 2 even numbers. In this case $[i]_{A_n}$ is Σ_1^0 .

The basic idea of this argument is the following lemma:

Lemma 4.4. *Suppose that f is a reduction of A_n to A_m . Then there is a computable function g so that for every k , if i is a Proper k -Coding index, then $f(i)$ is a Proper $g(k)$ -coding index.*

Proof. Since the only classes which are properly Σ_2^0 are those of proper l -coding indices, we see that if i is a Proper k -Coding index, then $f(i)$ is a Proper l -coding index for some l . We need only show that l depends only on k .

Suppose toward a contradiction that i and j are both Proper k -Coding indices yet $f(i)$ is a Proper l -Coding index and $f(j)$ is a Proper l' -Coding index with $l' \neq l$. By the Recursion Theorem, we build an c.e. set with index e as follows: We put $2k$ into W_e and we wait to see $2m$ enter $W_{f(e)}$ for some m . If we later see 2ℓ enter $W_{f(e)}$, we extend W_e to equal W_j and end the construction. On the other hand, if we see some $2\ell'$ enter $W_{f(e)}$, we extend W_e to equal W_i and end the construction.

If we were to never see any $2m$ enter $W_{f(e)}$, then we have a Proper k -Coding index which is not sent to a Proper l -Coding index for any l , which we have already observed is impossible. In either of the other cases, we have ensured that $e A_n i$ if and only if $f(e) A_m f(i)$ contradicting f being a reduction. \square

We now build the ceers X_n to satisfy requirements:

$$\mathcal{R}_{n,m}^k : \varphi_n \text{ is not a reduction of } A_n \text{ to } A_m.$$

We fix a ceer E which has infinitely many infinite classes and begin the construction with every X_n being equivalent to $\oplus_{i \in \omega} E$.

The strategy to meet one $\mathcal{R}_{n,m}^k$ -requirement. Suppose that the p th requirement in terms of priority is $\mathcal{R}_{n,m}^k$. Choose a column $\omega^{[k]}$ which has never been mentioned in the construction (in particular $X_{n,s} \upharpoonright \omega^{[k]} = E$). We make $X_{n,s+1} \upharpoonright \omega^{[k]} = \text{Id}_{p+2}$, and we restrain lower priority requirements from further collapsing $X_n \upharpoonright \omega^{[k]}$. We choose a number K which is a proper k -coding index (i.e., we choose an index for the set $\{2k\}$). We wait to see $g(k)$ converge. That is, we wait to see $2l$ enter $W_{f(K)}$ for some l . Once we see this l , there are two cases: If $X_m \upharpoonright \omega^{[l]}$ is restrained by a higher priority requirement, then $X_m \upharpoonright \omega^{[l]}$ already has fewer than $p+2$ classes. Otherwise, it is not restrained and we can make $X_m \upharpoonright \omega^{[l]} = \text{Id}_1$. In either case, we have satisfied the requirement since, were φ_n to be a reduction of A_n to A_m , then each of the $p+2$ non-equivalent proper k -coding indices in A_n would be sent to proper l -coding indices in A_m , but this cannot be injective on classes by the pigeon-hole principle (we use $p+2$ to ensure that $p+2 > 1$ even when $p=0$). In any case, whenever we act, we injure all lower priority requirements. These strategies fit together as a standard finite-injury construction. \square

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