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**SPECTRA OF RECURSIVE MODELS OF  
DISINTEGRATED STRONGLY MINIMAL THEORIES**

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ABSTRACT. This paper gives an overview of results about the spectra of computable (or recursive) models of strongly minimal disintegrated theories. Spectra of strongly minimal disintegrated theories in finite languages and in languages consisting of binary relation symbols will be described.

1. INTRO TO RECURSIVE MODEL THEORY

We assume that all languages  $L$  are countable and recursive. An  $L$ -structure  $A$  is *recursive* if  $|A| = \omega$  and the atomic diagram of  $A$  is recursive. An  $L$ -structure  $A$  is *decidable* if  $|A| = \omega$  and the elementary diagram of  $A$  is recursive.  $A$  is recursively (decidably) presentable if  $A$  is isomorphic to a recursive (decidable) model.

If  $B \subseteq A$ ,  $A$  is recursive, and the universe of  $B$  is a  $\Sigma_1$  subset of  $\omega$ , then  $B$  is recursively presentable.

**Definition 1.** *A first order theory  $T$  is strongly minimal if every definable subset of every model is a finite or co-finite subset of the model.*

Example: a regular acyclic graph with finite valence (say, the theory of a Cayley graph of a finitely generated group); a vector space (say, the theory of  $(\mathbb{Q}; +)$ ); an algebraically closed field (say, the theory of  $(\mathbb{C}; +; \cdot; 0; 1)$ ).

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**Definition 2.** For elements  $\bar{a}, b \in M$ , we say  $b \in \text{acl}(\bar{a})$  if there is a formula  $\phi(x; \bar{y})$  so that  $\phi(M; \bar{a})$  is finite and  $M \models \phi(b; \bar{a})$ , i.e.,  $b$  is in a finite  $\bar{a}$ -definable set.

A set  $S \subset M$  is *inter-algebraic* if for every  $a, b \in S$ ,  $a \in \text{acl}(b)$ .

A set  $S \subset M$  is *independent* if each  $x \in S$  is not in  $\text{acl}(S \setminus \{x\})$ .

If  $M$  is a model, a maximal independent subset is called a basis for  $M$ . The dimension of a set  $X$  is the size of a maximal independent subset. In strongly minimal theories, the above are well-defined. That is, if  $B_1$  and  $B_2$  are maximal independent subsets of  $X$ , then  $|B_1| = |B_2|$ .

If  $S$  is inter-algebraic, then the dimension of  $S$  is 1.

**Definition 3.** Let  $\phi(\bar{x})$  be a formula and  $T$  a strongly minimal theory. Then the Morley rank of  $\phi(\bar{x})$  is the maximal dimension of a tuple  $\bar{a} \in M \models T$  so that  $M \models \phi(\bar{a})$ .

Consider the formula  $x + x = y$  in the theory of  $\mathbb{Q}$ -vector spaces. This formula has Morley rank 1. The formula  $x + y = z$  has Morley rank 2. In each of our examples, the notion of dimension characterizes models. This is not a coincidence.

Baldwin and Lachlan [1] proved that if  $T$  is strongly minimal<sup>1</sup>, then each model of  $T$  is determined by its dimension. Moreover, if  $M$  is countable, then  $\dim(M) \in \{0, 1, \dots, \aleph_0\}$ .

Zil'ber conjectured that in fact our canonical examples of strongly minimal theories formed an exhaustive list. Zil'ber conjectured that every strongly minimal theory was of one of three types: disintegrated (essentially binary); locally modular (essentially a vector space); field-like (essentially an algebraically closed field). But Hrushovski [2] shown that the Zil'ber trichotomy is false. There are Hrushovski constructions which build non-trichotomous theories.

These structures are inherently combinatorial in nature, and have no algebraic content.

**Definition 4.** A strongly minimal theory is *disintegrated* if for every set  $S$ ,  $\text{acl}(S) = \bigcup_{a \in S} \text{acl}(\{a\})$ .

In the paper [3] it was proved the following

**Theorem 1.** If  $M \models T$  and  $T$  is a disintegrated strongly minimal theory, then the elementary diagram of  $M$  is model complete.

<sup>1</sup>They actually showed the result for  $\aleph_1$ -categorical theories, but I will talk only about strongly minimal theories

So, over enough parameters, every formula is equivalent to an  $\exists$ -formula, but not without parameters (see the example  $(\mathbb{N}, S)$ ).

**Corollary 1.** *If  $M \models T$  and  $T$  is a disintegrated strongly minimal theory, and  $X \subseteq M^n$  is any definable set, then  $X$  is recursive.*

It is possible for some models of a theory to be recursively presentable while some others are not.

Baldwin and Lachlan [1] showed that if  $T$  is strongly minimal, then each model of  $T$  is determined by a single cardinal invariant, its dimension. Moreover, if  $M$  is countable, then  $\dim(M) \in \omega + 1$ , and if  $T$  is not  $\aleph_0$  categorical, its countable models form a chain

$$M_0 \preceq M_1 \preceq M_2 \preceq \dots \preceq M_\omega.$$

If  $T$  is strongly minimal, define  $\text{SRM}(T) = \{k \in \omega + 1 \mid M_k \text{ is recursively presentable}\}$ .

## 2. THE SPECTRUM PROBLEM

The following problem was posed in the paper [4].

**Question 1.** *Which subsets of  $\omega + 1$  are spectra? In other words, for which  $S \subseteq \omega + 1$  does there exist some  $T$  so that  $\text{SRM}(T) = S$ ?*

The following sets are known to be spectra: no models; all models;  $\{0\}$  (Goncharov [4]);  $\{0, 1, \dots, n\}$  (Kudaibergenov [5]);  $\{0, 1, \dots\}$  (Khoussainov, Nies, Shore [6]);  $\{1, 2, \dots, \aleph_0\}$  (Khoussainov, Nies, Shore [6]);  $\{1\}$  (Nies [7]);  $\{1, 2, \dots, n\}$  (Nies, Hirschfeldt unpublished);  $\{1, 2, \dots\}$  (Nies, Hirschfeldt unpublished);  $\{\aleph_0\}$  (Hirschfeldt, Khoussainov, Semukhin, [8]);  $\{0, \aleph_0\}$  (Andrews [9]).

Also of interest is the following

**Question 2** (Lempp). *Which subsets of  $\omega + 1$  are spectra of theories in finite languages?*

The following sets are known to be spectra in finite languages: no models; all models;  $\{0\}$  (Herwig, Lempp, Ziegler [10]);  $\{0, 1, \dots, n\}$  (Andrews [11]);  $\{0, 1, \dots\}$  (Andrews [11]);  $\{\aleph_0\}$  (Andrews [11]).

For these results, we needed a Hrushvski construction, while each answer on question 1 (aside from  $\{0, \omega\}$ ) and the result from [10] was in a disintegrated theory.

**Question 3.** *What are the spectra of disintegrated theories in finite languages?*

The answer to this question was obtained in paper [12].

**Theorem 2.** *If  $T$  is strongly minimal and disintegrated in a finite language, then either every model of  $T$  is recursive, no model of  $T$  is recursive, or only the prime model of  $T$  is recursive.*

The remainder of this section will be dedicated to discussing this theorem. In next section this question will be discussed in infinite languages.

**Lemma 1.** *If  $T$  is disintegrated and strongly minimal in a finite language  $L$ , then there is a strongly minimal theory  $T'$  in a finite language  $L'$  comprised of only Rank 1 and Rank 0 relation symbols so that  $T$  and  $T'$  are interdefinable.*

*Sketch of the proof.* Take a relation in our language  $R(\bar{x})$ . Let  $k = \text{MR}(R)$ . We may assume  $R$  cannot be split into 2 pieces of the same Morley rank.

Let  $\bar{a}$  be a tuple of maximal dimension in a model so that  $M \models R(\bar{a})$ . Then  $\bar{a} = \bar{a}_1 \cup \dots \cup \bar{a}_n$  where each  $\bar{a}_i$  is inter-algebraic (every element is algebraic over every other element). Let  $\psi_i(\bar{a}_i)$  be a formula witnessing this inter-algebraicity, and let  $S(\bar{x})$  be the statement  $\bigwedge_i \psi_i(\bar{a}_i)$ . Then  $S$  has Morley rank  $k$ .

Since  $S \wedge \neg R$  and  $R \wedge \neg S$  both have smaller Morley Rank (irreducibility of  $R$ ), we remove  $R$  from our language and replace it with  $\{\psi_i\} \cup \{S \wedge \neg R, R \wedge \neg S\}$ . Each symbol has lower Morley rank. This procedure stops when all symbols have rank 0 or 1.  $\square$

But we need the reduction to respect the recursiveness of the models!

**Lemma 2.** *If  $T$  is disintegrated and strongly minimal in a finite language  $L$ , then there is a strongly minimal theory  $T'$  in a finite language  $L'$  comprised of only Rank 1 and Rank 0 relation symbols so that  $T$  and  $T'$  are  $\Delta_1$ -interdefinable (i.e.,  $T$  and  $T'$  are interdefinable and recursive models of  $T$  are recursive models of  $T'$  and vice versa).*

*Proof.* This is just the last lemma plus the corollary 1.  $\square$

We note that here we very strongly use the fact that the language is finite. Otherwise, you would need some uniformity in that corollary, which we do not have.

We have now reduced to the case where every symbol in the language is Rank 1 or Rank 0. We still need to show that either every model, no model, or only the prime model is recursive.

**Lemma 3.** *If  $T$  is disintegrated strongly minimal in a finite language comprised of all Rank 1 and Rank 0 relation symbols, then either every model, no model, or only the prime model is recursive.*

The content of this proof will be incredibly important for what follows, as it truly analyzes the structure of the models. It shows us how to picture models of a general disintegrated strongly minimal theory. Much of the analysis first appears in the [10] paper.

Given a relation symbol  $R(\bar{x})$  of rank 1, define the relations

$$(R, i, j)(a, b) := \exists \bar{x} R(\bar{x}) \wedge x_i = a \wedge x_j = b.$$

For a generic  $a$  (i.e.  $a \notin \text{acl}(\emptyset)$ ), there are only finitely many  $b$  so that  $(R, i, j)(a, b)$  or  $(R, i, j)(b, a)$ . Fix the finite sets of elements

$$A_{R,i,j} := \{x \mid \exists^\infty y (R, i, j)(x, y)\}; \quad B_{R,i,j} := \{x \mid \exists^\infty y (R, i, j)(y, x)\}.$$

Define  $E_{R,i,j}(a, b) := (R, i, j)(a, b) \wedge a \notin A_{R,i,j} \wedge b \notin B_{R,i,j}$  or  $\neg(R, i, j)(a, b) \wedge (a \in A_{R,i,j} \vee b \in B_{R,i,j})$ . Note that if  $E_{R,i,j}(a, b)$  then  $a, b$  is inter-algebraic.

**Definition 5.** *For any element  $a \in M \models T$ , define  $\text{Nbh}_s(a)$  inductively as follows  $\text{Nbh}_0(a) = \{a\}$ ; if  $b \in \text{Nbh}_s(a)$  and  $E_{R,i,j}(b, c)$  or  $E_{R,i,j}(c, b)$ , then  $c \in \text{Nbh}_{s+1}(a)$ . We define  $\text{Nbh}(a) = \bigcup_{s \in \omega} \text{Nbh}_s(a)$ .*

**Lemma 4.** *If  $a, b \in M$  and  $X \subset M$ , then  $\text{Nbh}(a) \cong \text{Nbh}(b)$  over  $X$  if and only if  $(\forall s)(\text{Nbh}_s(a) \cong \text{Nbh}_s(b) \text{ over } X)$ .*

*Proof.* It is following from König's Lemma. □

**Lemma 5.** *Fix  $a \notin \text{acl}(\emptyset)$ . Then the set of elements inter-algebraic with  $a$  is exactly  $\text{Nbh}(a)$ .*

*Proof.* Build isomorphisms. □

**Lemma 6.** *Fix  $a \in M$  not in  $\text{acl}(\emptyset)$ . For any  $b \in M$ ,  $b \notin \text{acl}(\emptyset)$  if and only if  $\text{Nbh}(a) \cong \text{Nbh}(b)$  over  $B := \bigcup_{R,i,j} (A_{R,i,j} \cup B_{R,i,j})$ .*

*Proof.* Build isomorphisms. □

*Finishing the proof Theorem 2.* Fix a recursive model  $M$  of dimension  $> 0$ . Every  $E_{R,i,j}$  is recursive by the Corollary 1. So,  $\text{Nbh}_i(x)$  is uniformly recursive for  $x \in M$ . Thus  $\text{Nbh}(x)$  is uniformly recursively enumerable.

Let us prove that  $\text{acl}(\emptyset)$  is recursively enumerable. Fix  $a \notin \text{acl}(\emptyset)$ . Then  $x \in \text{acl}(\emptyset)$  if and only if  $\exists i \text{ Nbh}_i(x) \not\cong \text{Nbh}_i(a)$  over  $B$ . This is  $\Sigma_1^0$ .

Now the  $k$ -dimensional model is comprised of a copy of  $\text{acl}(\emptyset)$  and  $k$  copies of  $\text{Nbh}(a)$ . This makes the model either a recursively enumerable subset of  $M$  or a recursive expansion of  $M$ .

## 3. THE SPECTRUM PROBLEM

This section will be dedicated to discussing of the next

**Theorem 3** ([13]). *If  $T$  is strongly minimal and the language is comprised of binary relation symbols, then  $SRM(T)$  is one of the following seven:  $\emptyset$ ,  $[0, \omega]$ ,  $\{0\}$ ,  $\{1\}$ ,  $\{0, 1\}$ ,  $\{\omega\}$ ,  $[1, \omega]$ .*

**Lemma 7.** *If  $T$  has a recursive model of dimension  $\geq 2$ , then  $T$  is  $\Delta_1$ -interdefinable with a theory  $T'$  so that the language of  $T'$  is comprised of relation symbols of rank  $\leq 1$ .*

*Proof.* Fix  $a, b \in M \models T$  where  $a, b$  are independent and  $M$  is the recursive model of dimension  $\geq 2$ . Then  $R(x, y)$  has Morley rank 2 if and only if  $M \models R(a, b)$ . So, if  $M \models R(a, b)$ , replace  $R$  by  $\neg R$ , which has Morley rank  $\leq 1$ .  $\square$

Note that if  $T$  has no recursive model of dimension  $\geq 2$ , then  $SRM(T) \subseteq \{0, 1\}$ , and all 4 such sets are spectra of disintegrated strongly minimal theories in binary languages.

For each relation  $R_i(x, y)$  denote  $E_i(a, b) := E_{R_i, 0, 1}(a, b)$ . Note that  $E_i(a, b)$  implies that  $a, b$  are inter-algebraic.

**Definition 6.** *Define the neighborhoods  $Nbh_k(a)$  inductively as follows  $Nbh_0(a) = \{a\}$ ; if  $b \in Nbh_k(a)$  and  $E_i(b, c)$  for  $i \leq k + 1$ , then  $c \in Nbh_{k+1}(a)$ ; we define  $Nbh(a) = \bigcup_k Nbh_k(a)$ .*

Note that (essentially) the same argument as last time shows that lemma 5 is true in this case (i.e. if  $a \notin \text{acl}(\emptyset)$  then the set of elements inter-algebraic with  $a$  is exactly  $Nbh(a)$ ).

Fix  $M$  a recursive model of dimension  $\geq 2$ .

**Lemma 8.** *For any  $a \in M$ ,  $Nbh(a)$  is a recursively enumerable set.*

*Proof.* The definition is  $\Sigma_1^0$  over the definition of  $E_i$ , which is recursive, except for deciding  $A_i := \{a \in M \mid \exists^\infty z R_i(a, z)\}$  and  $B_i := \{b \in M \mid \exists^\infty z R_i(z, b)\}$ .

Fix independent elements  $c_1, c_2 \in M$ . Then  $x \in A_i$  if and only if  $R_i(x, c_1) \wedge R_i(x, c_2)$ . Similarly for  $B_i$ . Thus,  $A_i$  and  $B_i$  are uniformly recursive in  $i$ , and so  $Nbh_k(a)$  is uniformly recursive, and  $Nbh(a)$  is recursively enumerable.  $\square$

Note that if  $\text{acl}(\emptyset)$  were also recursively enumerable, we would be able to conclude like last time, but that's not true anymore.

**Lemma 9.** *Let  $N$  be recursive strongly minimal structure. Let  $M \preceq N$  be a  $\Delta_{\epsilon}^0$  subset of  $N$ . Let  $A \subseteq M$  be any infinite recursively enumerable set. Then there is a recursive copy of  $M$ .*

*Proof.* Watch the  $\Delta_{\epsilon}^0$  approximation of  $\chi_M(b)$  for elements  $b \in N$ . When the approximation puts  $b \in M$ , we copy it into our structure. When it puts  $b \notin M$ , we need to "un-copy" it. That is, we reassign it to being a copy of an element in  $A$ . What do we need to verify? Let  $M_0$  be the rest of the elements that we believe to be in  $M$ . Let  $\phi(M_0, b)$  be the quantifier-free statement that we have committed to so far. We need to check that for some element  $a \in A$ ,  $M \models \phi(M_0, a)$ . Strong minimality gives us that either  $b \in \text{acl}(M_0)$ , which implies  $b \in M$  – so the approximation changes back, or there is such an  $a$ .  $\square$

**Lemma 10.** *If  $T$  has a recursive model of finite dimension  $\geq 2$ , then every model of dimension  $k \in [1; \omega]$  is recursively presentable.*

*Proof.* Fix  $N$  that model of finite dimension  $k \geq 2$ . First we show the result for  $M$  smaller than  $N$ .

Let  $M \preceq N$  be a model of dimension  $\geq 1$ . Let  $A = \text{Nbh}(a)$  for an element  $a \in M \setminus \text{acl}(\emptyset)$ . As each  $\text{Nbh}(b)$  is  $\Sigma_1^0$ ,  $M$  is a union of a  $\Pi_1^0$  set ( $\text{acl}(\emptyset)$ ) and a  $\Sigma_1^0$  set ( $\bigcup \text{Nbh}(b)$  where the union ranges over all  $b$  in a basis for  $M$ ), thus  $M$  is  $\Delta_2^0$ . If  $A$  is infinite, we are done. If  $A$  is finite, we are done, since  $M$  is a co-finite subset of  $N$ , thus recursive.

Now, let us consider  $N \preceq M$ . Let us show that the model of dimension  $k + 1$  is recursively presentable.

Let  $C$  be the structure  $\text{Nbh}(a)$  for  $a \in N \setminus \text{acl}(\emptyset)$ .  $C$  is a recursive structure.

Take a copy of  $N \cup C$ . Let  $R(c, d)$  hold for  $c \in C$ ,  $d \in N$  if and only if  $d \models \exists^\infty z R(z, d)$  which holds if and only if  $R(c_1, d) \wedge R(c_2, d)$  for a fixed independent pair  $c_1, c_2 \in N$ . Similarly, let  $R(d, c)$  hold if and only if  $R(d, c_1) \wedge R(d, c_2)$ . This gives a recursive presentation of the  $k + 1$ -dimensional model. By inducting, we get every finite dimensional model. As this extension is uniform, we get  $M_\omega$  recursively as well.  $\square$

**Corollary 2.** *If  $T$  is a disintegrated strongly minimal theory with binary language, then  $\text{SRM}(T)$  is one of the following 10:  $\emptyset$ ,  $[0, \omega]$ ,  $\{0\}$ ,  $\{1\}$ ,  $\{0, 1\}$ ,  $\{\omega\}$ ,  $[1, \omega]$  OR  $\{0, \omega\}$ ,  $\{1, \omega\}$ ,  $\{0, 1, \omega\}$ .*

We need to show that the last three are not spectra of disintegrated strongly minimal theories with binary languages.

**Lemma 11.** *If  $0, \omega \in \text{SRM}(T)$  or  $1, \omega \in \text{SRM}(T)$ , then  $2 \in \text{SRM}(T)$  (and thus  $[1, \omega] \subseteq \text{SRM}(T)$ ).*

*Proof.* There are two cases.

*Case 1.* For every  $c \in \text{acl}(\emptyset)$ ,  $c \in \bigcup_i (A_i \cup B_i)$  or there is an  $i$  and a  $d \in A_i \cup B_i$  so  $E_i(d, c)$ . Then  $\text{acl}(\emptyset)$  is  $\Sigma_1^0$  in  $M_\omega$ , and so every  $M_k$  is a  $\Sigma_1^0$  subset of  $M_\omega$ .

*Case 2.* There is an element  $c$  so  $c \notin \bigcup_i (A_i \cup B_i)$  and  $c$  is not  $E_i$  connected to any elements of  $A_i \cup B_i$  for each  $i$ . There are two subcases.

*Case 2A.* There is no element  $d$  so that for every  $x$ ,  $\exists^\infty z R(x, z)$  if and only if  $R(x, c) \wedge R(x, d)$ . In this case, again  $\text{acl}_{M_\omega}(\emptyset)$  is  $\Sigma_1^0$ . Again, every  $M_k$  is a  $\Sigma_1^0$  subset of  $M_\omega$ , thus is recursive.

*Case 2B.* There is  $d \in \text{acl}(\emptyset)$  so for every  $x$ ,  $\exists^\infty z R(x, z)$  if and only if  $R(x, c) \wedge R(x, d)$ .

Let  $C = \text{Nbh}(a)$  for  $a \in M_\omega \setminus \text{acl}(\emptyset)$ . Then take  $M_0$  (or  $M_1$ ) and append it by 2 (or 1) copies of  $C$ . Define  $R(a, b)$  to hold where  $a \in M_0$  and  $b \in C$  if and only if  $R(a, c) \wedge R(a, d)$ . This gives a recursive copy of  $M_2$ .  $\square$

This lemma finishes the proof of the theorem 3.

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