

THE THEORY OF CEERS COMPUTES TRUE ARITHMETIC

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ABSTRACT. We show that the theory of the partial order of computably enumerable equivalence relations (ceers) under computable reduction is 1-equivalent to true arithmetic. We show the same result for the structure comprised of the dark ceers and the structure comprised of the light ceers. We also show the same for the structure of \mathcal{I} -degrees in the dark, light, or complete structure. In each case, we show that there is an interpretable copy of $(\mathbb{N}, +, \cdot)$.

1. INTRODUCTION

A major theme in investigating computability theoretic reducibilities has been to measure, and when possible to characterize, the complexity of the first order theory of their degree structures. Throughout the paper we regard a degree structure as a poset, and if \mathcal{P} is a poset then the *theory of \mathcal{P}* , denoted by $\text{Th}(\mathcal{P})$, is the set of sentences, in the first order language of posets, that are true in \mathcal{P} . Investigating the theory of a degree structure is an important task, not only because it sheds relevant information about the corresponding reducibility, but also because it generally stimulates useful techniques and constructions which are developed for this purpose.

Typically, a reducibility is a binary relation \leq on subsets of the set ω of natural numbers, which gives rise to a degree structure \mathcal{D} which partitions the power set $P(\omega)$ into equivalence classes called degrees. Let $\nu : P(\omega) \rightarrow \mathcal{D}$ be the surjection so that $\nu(X)$ is the degree of X , and \leq denote the partial ordering relation on \mathcal{D} . The relations (in X, Y) $\nu(X) = \nu(Y)$ and $\nu(X) \leq \nu(Y)$ are typically arithmetical, so that one can effectively translate first order sentences regarding degrees to second order sentences of arithmetic. This yields a reduction $\text{Th}(\mathcal{D}) \leq_1 \text{Th}^2(\mathbb{N})$, where the latter symbol denotes true second order arithmetic. In many cases, it is true that these two theories are 1-equivalent, and the challenge is to show that the reverse reduction, i.e. $\text{Th}^2(\mathbb{N}) \leq_1 \text{Th}(\mathcal{D})$, holds as well. By the Myhill Isomorphism Theorem, the two theories are then computably isomorphic, and thus $\text{Th}(\mathcal{D})$ is as complicated as it can be. The literature here is indeed rich of classical and celebrated results, starting from Simpson [29] who showed that the theory of the Turing degrees is computably isomorphic to $\text{Th}^2(\mathbb{N})$. See also [30]. To mention two other major reducibilities, the theory of the m -degrees ([22]) and the theory of the enumeration degrees ([31], see also [7]) are also computably isomorphic to $\text{Th}^2(\mathbb{N})$.

When no restriction is taken on the universe of the reducibility, then one talks about the *global* degree structure of that reducibility. It is common however to consider *local* degree structures as

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well, by restricting attention to special countable families of degrees. This is the case for instance (just to consider some local structures of the aforementioned global structures) of the Turing degrees below the first jump, or the Turing degrees of the computably enumerable (c.e.) sets, or the m -degrees of the c.e. sets, or the enumeration degrees below the first enumeration jump. If \mathcal{D} is a local structure then typically one finds a surjection $\nu : \omega \rightarrow \mathcal{D}$ such that the relations (in x, y) $\nu(x) = \nu(y)$ and $\nu(x) \leq \nu(y)$ are arithmetical, so that one can effectively translate sentences on \mathcal{D} into first order arithmetical sentences, establishing a reduction $\text{Th}(\mathcal{D}) \leq_1 \text{Th}^1(\mathbb{N})$, where the latter set denotes true first order arithmetic $\text{Th}(\mathbb{N}, +, \times)$. To show that $\text{Th}(\mathcal{D})$ is as complicated as possible (i.e. computably isomorphic to $\text{Th}^1(\mathbb{N})$) it is then enough to show $\text{Th}^1(\mathbb{N}) \leq_1 \text{Th}(\mathcal{D})$. For instance, this has been done for the aforementioned local structures: for the Turing degrees below the first jump, see Shore [28], for the c.e. Turing degrees see Nies, Shore and Slaman [25], for the c.e. m -degrees, see Nies [23], for the enumeration degrees below the first enumeration jump, see Ganchev and Soskova [14, 15].

The above examples are about reducibilities on sets of natural numbers. We consider in this paper a reducibility on equivalence relations on ω , instead of subsets of ω . The reduction is defined as follows: if R, S are equivalence relations on ω , we say that R is *computably reducible* (or, simply, *reducible*) to S (notation: $R \leq S$) if there is a computable total function f such that

$$(\forall x, y)[x R y \Leftrightarrow f(x) S f(y)].$$

As with other reducibilities, \leq gives rise to an equivalence relation \equiv , where $R \equiv S$ if $R \leq S$ and $S \leq R$. The equivalence class of an equivalence relation R under \equiv will be called the *degree of R* . The first study of computable reducibility on equivalence relations on natural numbers was initiated by Ershov [9, 10] in the 1970s. Recently, there has been a revived interest in this reducibility, motivated by considering computable reducibility as an effective version of Borel reducibility on equivalence relations [12]. Borel reducibility on equivalence relations is a primary target of interest in descriptive set theory, see for instance [5]. The revived interest in computable reducibility has also been caused by its high potential as a tool for measuring the computational complexity of classification problems in computable mathematics. For instance it is shown in [11] that the isomorphism relation for various familiar classes of computable groups is Σ_1^1 -complete under this reducibility.

The global structure of the degrees of equivalence relations, however, has not been extensively studied. Much more attention has been given to its local structure consisting of the degrees of the c.e. equivalence relations (commonly called *ceers* after [16]). Indeed, ceers have played a leading role in the tale of computable reducibility as they appeared as the main characters of what are perhaps the first results about \leq (although before the notion appeared in the literature). Namely, Miller III [20] constructed a finitely presented group whose word problem is Σ_1^0 -complete with respect to \leq , and Miller III [20] proved that the isomorphism problem for finitely presented groups is Σ_1^0 -complete with respect to \leq . For other applications of \leq to word problems of finitely presented groups see [26]. The first work explicitly tackling \leq on ceers was done by Ershov [9], who pointed out important examples of Σ_1^0 -complete ceers, showing also that their degree is join-irreducible. In the 1980s, the reducibility \leq on ceers was applied to study computability theoretic properties of the relation of provable equivalence of sufficiently expressive formal systems, see for instance [6, 21]. Additional interest for computable reducibility on ceers comes from the study of c.e. presentations of structures, as is examined for instance in [13, 17]. It is also worth noticing that ceers have been investigated in computability theory also not in connection with computable reducibility. For instance, Carroll [8] studied the lattice of ceers under inclusion, and Nies [24] studied ceers

modulo finite differences. They both showed that the first order theory of the resulting structures is computably isomorphic to $\text{Th}^1(\mathbb{N})$.

More explicitly oriented toward a degree-theoretic approach are the papers on ceers by Gao and Gerdes [16], Andrews et al. [2], [3], and finally Andrews and Sorbi [4]. The last paper provides a thorough investigation of the structure **Ceers** comprised of \equiv -degrees of ceers under the reducibility \leq , with emphasis on existence and non-existence of meets and joins, minimal covers, definable classes of degrees, and automorphisms.

It was shown in [2] that $\text{Th}(\mathbf{Ceers})$ is undecidable, and even the $\forall\exists\forall$ -fragment (again, in the language of posets) is undecidable. In this paper we completely characterize the complexity of $\text{Th}(\mathbf{Ceers})$ by showing that it is in fact as complicated as it can be, namely computably isomorphic to $\text{Th}^1(\mathbb{N})$. We do this also for two suborders of **Ceers** introduced in [4], called **Light** and **Dark** which are defined in the next section, and also for the quotient structures obtained by quotienting the three structures **Ceers**, **Light**, and **Dark** modulo uniform joins with finite ceers. In each case, we show that there is an interpretable copy of $(\mathbb{N}, +, \cdot)$ in the degree structure.

1.1. Outline of proof. In section 4, we give a definable copy of $(\mathbb{N}, +, \cdot)$ using a parameter in **Dark**/ \mathcal{I} . To do this, we show that we can encode any graph (modulo some isolated vertices) in a particular way. In particular, for any **Dark**/ \mathcal{I} -degree \mathbf{c} , we let the universe of $G_{\mathbf{c}}$ be the collection of minimal \mathcal{I} -degrees $\leq \mathbf{c}$. We say that \mathbf{a} is an \mathcal{I} -strongly minimal cover of a pair of incomparable \mathcal{I} -degrees \mathbf{d} and \mathbf{e} if the degrees $<_{\mathcal{I}} \mathbf{a}$ are precisely the degrees $\leq_{\mathcal{I}} \mathbf{d}$ and the degrees $\leq_{\mathcal{I}} \mathbf{e}$. We put an edge in $G_{\mathbf{c}}$ between degrees \mathbf{d} and \mathbf{e} if there are two incomparable degrees $\leq \mathbf{c}$ which are \mathcal{I} -strongly minimal covers of \mathbf{d} and \mathbf{e} . Section 3 is devoted to proving technical details about minimal \mathcal{I} -degrees and their \mathcal{I} -strongly minimal covers. The main purpose of these details are to allow us in section 4 to construct, for any given computable graph G , a dark \mathcal{I} -degree \mathbf{c} so that $G_{\mathbf{c}}$ is isomorphic to a disjoint of G with a graph that has no edges. In particular, since countable graphs with no isolated vertices are universal for all countable structures under definability, this gives us a coded copy of $(\mathbb{N}, +, \cdot)$ using a single parameter \mathbf{c} .

In section 5, we remove the dependence on the parameter to define a copy of $(\mathbb{N}, +, \cdot)$. We do this as follows: We fix an interpretation of $(\mathbb{N}, +, \cdot)$ in a graph. We first isolate the collection of degrees \mathbf{c} for which $G_{\mathbf{c}}$ interprets (via this fixed interpretation) a model of Robinson's system Q of arithmetic. We call these good codes. Next, we consider the collection of pairs (\mathbf{c}, \mathbf{d}) where \mathbf{c} is a good code and \mathbf{d} is in \mathbf{c} 's encoded model of Q . Our goal is to isolate the collection of such pairs where \mathbf{d} is in the standard part of the model. To do this, we define an encoding of functions inside **Dark**/ \mathcal{I} (via our encoding of graphs) and define an equivalence relation on such pairs $(\mathbf{c}, \mathbf{d}) \sim (\mathbf{c}', \mathbf{d}')$ if and only if there is an encoded order-preserving bijection between the interval $[\mathbf{0}, \mathbf{d}]$ in \mathbf{c} 's model of Q and the interval $[\mathbf{0}, \mathbf{d}']$ in \mathbf{c}' 's model of Q . In other words, \mathbf{d} and \mathbf{d}' code the same number in their respective models. Not every function can be encoded in the structure, since it is countable, but our encoding is sufficient to capture every finite function on our models of Q . Much of section 5 is devoted to this encoding. Finally, we define \mathcal{N} to be the collection of pairs (\mathbf{c}, \mathbf{d}) where \mathbf{c} is a good code and for every good code \mathbf{c}' , there is some \mathbf{d}' so that $(\mathbf{c}, \mathbf{d}) \sim (\mathbf{c}', \mathbf{d}')$. Since there is some good code which encodes a copy of $(\mathbb{N}, +, \cdot)$, and our collection of encoded functions includes all the finite functions on our models of Q , this isolates the \sim -classes of the standard naturals within the various models of Q . Finally, we can define the operations $+$ and \cdot on \mathcal{N} which makes it isomorphic to $(\mathbb{N}, +, \cdot)$.

The above gives an interpreted copy of $(\mathbb{N}, +, \cdot)$ in **Dark**/ \mathcal{I} . Since \mathcal{I} -equivalence is \emptyset -definable (i.e., definable without parameters) in **Dark**, and **Dark** is \emptyset -definable in **Ceers**, this shows the result for **Dark**/ \mathcal{I} , **Dark**, and **Ceers**. We also here observe that the same encoding works in **Ceers**/ \mathcal{I} .

Note that in both the **Light**/ \mathcal{I} degrees and the **Light** degrees, there is a single minimal degree. Thus our encoding needs to be changed to consider **Light**/ \mathcal{I} -degrees which are minimal over the degree of Id . In section 6 we show that with this slightly altered encoding, we can again encode every computable graph modulo some isolated vertices in the **Light**/ \mathcal{I} -degrees using a parameter. In section 7, we repeat the strategy used in section 5 to remove the dependence on the parameter in the **Light**/ \mathcal{I} -degrees. In this setting, the simple encoding of functions used in section 5 does not suffice, so we introduce a more complicated encoding of functions. Aside from the different encoding of functions, the strategy is the same and again we encode our copy of $(\mathbb{N}, +, \cdot)$ in **Light**/ \mathcal{I} . Since \mathcal{I} -equivalence is \emptyset -definable in **Light**, this also yields the result for **Light**.

2. BACKGROUND MATERIAL

For more information and details about unexplained computability theoretic terminology or results exploited in the paper without any reference, the reader may consult any standard textbook, see e.g. [27, 32]. In this section we review some background material concerning ceers and computable reducibility. The \equiv -degree of an equivalence relation R will be denoted by $\text{deg}(R)$.

2.1. The classes \mathcal{I} , **Light, and **Dark**.** We recall the following partition of ceers, introduced and studied in [4]. Let R be a ceer:

- Let Id_n be the equivalence relation given by $x \text{Id}_n y$ if and only if $x \equiv y \pmod{n}$.
- Let Id be the equivalence relation given by $x \text{Id} y$ if and only if $x = y$.
- R is *finite* if it has only finitely many equivalence classes. Note that for each $n \geq 1$, R has exactly n classes if and only if $R \equiv \text{Id}_n$.
- R is *light* if there is an infinite c.e. set W such that $x \not R y$ for each pair of distinct $x, y \in W$ (following [16], a set with this property will be called a *transversal for R*). Equivalently, a ceer R is light if and only if $\text{Id} \leq R$.
- R is *dark* if it is neither finite nor light.

The symbols \mathcal{I} , **Light**, **Dark** denote the classes of finite ceers, light ceers, and dark ceers, respectively. These classes partition the ceers, and give rise to a corresponding partition of the degrees of ceers into three classes of degrees (still denoted by \mathcal{I} , **Light**, **Dark**). \mathcal{I} is an initial segment of **Ceers** having order type ω . In (\mathbf{Ceers}, \leq) , the degree of Id , and thus each of these three classes are first order \emptyset -definable [4, Corollary 8.1]. **Ceers**, **Light**, and **Dark** are neither upper nor lower semilattices. In this regard, the most spectacular case is provided by dark degrees, as no pair of incomparable dark degrees has either meet or join in **Ceers** or in **Dark** [4, Theorems 5.8 and 7.18].

2.2. Some general facts about ceers. We describe three constructions of new ceers starting from given ceers and/or c.e. sets.

The first construction is the *uniform join* $R \oplus S$ which is the equivalence relation which copies R on the even numbers and S on the odd numbers: $x R \oplus S y$ if there exist u, v such that $x = 2u, y = 2v$ and $u R v$, or $x = 2u + 1, y = 2v + 1$ and $u S v$. This operation extends in the obvious way to any countable number of equivalence relations, see Section 2.1 of [4].

The second construction is described in detail in Section 2.3 of [4]:

Definition 2.1. If E is a ceer and W is a non-empty c.e. set then $E \upharpoonright W$ (called the *restriction of E to W*) is the ceer $x E \upharpoonright W y$ if and only if $h(x) E h(y)$, where $h : \omega \rightarrow W$ is any computable surjection (up to \equiv the definition does not depend on the chosen h).

Remark 2.2. It is clear that h provides a reduction $E \upharpoonright W \leq E$, which we call the *inclusion of $E \upharpoonright W$ into E* . If $X \leq R$ via a reduction f then $X \equiv R \upharpoonright W$, where $W = \text{range}(f)$.

Fact 2.3. If $X \leq R_1 \oplus R_2$ then there are ceers $X_1 \leq R_1$ and $X_2 \leq R_2$ such that $X \equiv X_1 \oplus X_2$.

Proof. If f is a reduction for $X \leq R_1 \oplus R_2$ and $W = \text{range}(f)$ then $X \equiv (R_1 \oplus R_2) \upharpoonright W \equiv R_1 \upharpoonright V_1 \oplus R_2 \upharpoonright V_2$, where $V_1 = \{x \mid 2x \in W\}$ and $V_2 = \{x \mid 2x + 1 \in W\}$. \square

The third construction is described in the following definition.

Definition 2.4. Let $W \subseteq \omega^2$. We denote by $E_{/W}$ the equivalence relation generated by the set of pairs $E \cup W$. If W is a singleton, say $W = \{(x, y)\}$, then we simply write $E_{/(x,y)}$ instead of $E_{/\{(x,y)\}}$.

Notice that if W is c.e. and E is a ceer then $E_{/W}$ is a ceer as well.

We will also make use of the following easy facts about ceers:

Fact 2.5. *The following hold:*

- (1) For every pair S, T of ceers and $k \geq 0$, $S \equiv T$ if and only if $S \oplus \text{Id}_k \equiv T \oplus \text{Id}_k$ (where for any equivalence relation R , we let $R \oplus \text{Id}_0 = R$).
- (2) If f is a reduction from R to S which omits exactly k S -classes then $R \oplus \text{Id}_k \equiv S$.
- (3) If W is a non-empty c.e. set missing exactly k equivalence classes of a ceer R then $R \upharpoonright W \oplus \text{Id}_k \equiv R$.

Proof. The first item is essentially [4, Lemma 2.1]. The second item comes from Lemma 2.8 in [4]; the third item follows from the previous one and the fact that under the assumptions the inclusion reduction $R \upharpoonright W \leq R$ misses exactly k equivalence classes. \square

2.3. Reducibility modulo \mathcal{I} . We recall the following reducibility from [4].

Definition 2.6. We say $R \leq_{\mathcal{I}} S$ if there is some k so that $R \leq S \oplus \text{Id}_k$.

Define $R \equiv_{\mathcal{I}} S$ if $R \leq_{\mathcal{I}} S$ and $S \leq_{\mathcal{I}} R$. The equivalence class of R under $\equiv_{\mathcal{I}}$ will be denoted by $\text{deg}_{\mathcal{I}}(R)$ and called the \mathcal{I} -degree of R . $\mathbf{Ceers}_{/\mathcal{I}}$ is the collection of all $\text{deg}_{\mathcal{I}}(R)$, where R is a ceer.

Fact 2.7. If $X \leq_{\mathcal{I}} R_1 \oplus R_2$ then there are ceers $X_1 \leq R_1$ and $X_2 \leq R_2$ so that $X \equiv_{\mathcal{I}} X_1 \oplus X_2$.

Proof. There is an n so that $X \leq R_1 \oplus R_2 \oplus \text{Id}_n$. By Fact 2.3, this shows that there are $X_1 \leq R_1$ and $X_2 \leq R_2$ and $F \leq \text{Id}_n$ so that $X \equiv X_1 \oplus X_2 \oplus F$. Then $X \equiv_{\mathcal{I}} X_1 \oplus X_2$. \square

We consider the six structures: **Ceers**, **Dark**, **Light**, $\mathbf{Ceers}_{/\mathcal{I}}$, $\mathbf{Dark}_{/\mathcal{I}}$, $\mathbf{Light}_{/\mathcal{I}}$. For elementary differences between these classes, and for more on their structural properties, see [4]. We note:

Lemma 2.8 ([4], Obs. 9.7). *In each of **Ceers**, **Dark**, and **Light**, the equivalence relation $\equiv_{\mathcal{I}}$, and thus also the partial order $\leq_{\mathcal{I}}$, is \emptyset -definable.*

Notice also:

Lemma 2.9. *If \mathcal{D} is any of the six structures **Ceers**, **Dark**, **Light**, **Dark** $_{/\mathcal{I}}$, **Light** $_{/\mathcal{I}}$, and **Ceers** $_{/\mathcal{I}}$ then $\text{Th}(\mathcal{D}) \leq_1 \text{Th}^1(\mathbb{N})$.*

Proof. Let $\{R_z \mid z \in \omega\}$ be the indexing of ceers defined in [2], and let $\nu : \omega \rightarrow \mathbf{Ceers}$ be given by $\nu(x) = \deg(R_x)$. Then it is easy to see that the relations (in x, y) $\nu(x) = \nu(y)$ and $\nu(x) \leq \nu(y)$ are arithmetical, so that one can effectively translate sentences on degrees into first order arithmetical sentences, showing $\text{Th}(\mathbf{Ceers}) \leq_1 \text{Th}^1(\mathbb{N})$. Since each of the other five structures are interpretable without parameters in **Ceers** by [4, Corollary 8.1] and Lemma 2.8, this shows that their theories also are $\leq_1 \text{Th}^1(\mathbb{N})$. \square

For each of the structures of degrees of ceers mentioned in Lemma 2.9, we show that there is a copy of $(\mathbb{N}, +, \cdot)$ which is interpreted without parameters in the structure, and thus by Lemma 2.9, the theory is 1-equivalent to true arithmetic. In view of Lemma 2.8, to yield the result for all six structures we need only find an interpreted copy of $(\mathbb{N}, +, \cdot)$ in the three structures **Dark** $_{/\mathcal{I}}$, **Light** $_{/\mathcal{I}}$, and **Ceers** $_{/\mathcal{I}}$.

2.4. Self-full ceers. The following obvious facts about dark ceers hold:

Fact 2.10. *The following hold:*

- (1) *if S is not finite, $S \leq_{\mathcal{I}} T$ and T is dark then so is S ;*
- (2) *if R is dark and S is either finite or dark then $R \oplus S$ is dark;*
- (3) *if R is dark then so is any $R_{/W}$. In particular, if R, S are dark then so is any $(R \oplus S)_{/(2x, 2y+1)}$.*

Proof. Item (1) is [4, Observation 6.3]; item (2) is [4, Observation 3.2]; item (3) follows straight from the definitions. \square

Definition 2.11. A ceer R is self-full if $R \oplus \text{Id}_1 \not\leq R$. Equivalently ([4, Observation 4.2]), R is self-full if whenever $\varphi : R \rightarrow R$ is a reduction, φ is onto the classes of R , i.e., the range of φ intersects every R -class.

The following fact collects useful properties of self-full ceers and the $\oplus \text{Id}_1$ operation:

Fact 2.12. *The following hold:*

- (1) *For any ceers R and S , if $S < R \oplus \text{Id}_1$ then $S \leq R$. For any ceers R and S , if $R < S$ then $R \oplus \text{Id}_1 \leq S$.*
- (2) *Every dark ceer is self-full.*

Proof. Item (1) is [4, Lemma 4.5], and item (2) is [4, Lemma 4.6]. \square

It is also useful to note:

Observation 2.13. *If R is self-full and $R \equiv E$, then any reduction $\varphi : R \rightarrow E$ must be onto the classes of E .*

Proof. Consider the pair of reductions $R \leq E \leq R$. If we were to have a reduction of R into E which is not onto the classes of E , then the composition would be a reduction of R to itself which is not onto the classes of R , contradicting R being self-full. \square

3. MINIMAL CLASSES IN $\mathbf{Dark}_{/\mathcal{I}}$

We now proceed with examining some facts about dark minimal \mathcal{I} -degrees, on which we will code models of arithmetic. There are two types of minimal degrees in $\mathbf{Dark}_{/\mathcal{I}}$. The first is the \mathcal{I} -degree of a ceer D which is a dark minimal ceer. The second is an \mathcal{I} -degree which, as a class of \equiv -degrees, has the order type of \mathbb{Z} and bounds no other non-zero \mathcal{I} -degree. Indeed, suppose that R is a dark ceer, and $\deg_{\mathcal{I}}(R)$ is minimal. First of all notice that the ceers $\{R_k \mid k \in \omega\}$, where $R_k = R \oplus \text{Id}_k$, give rise, as k strictly increases, to a strictly increasing sequence of \equiv -degrees within $\deg_{\mathcal{I}}(R)$ (where we let $R_0 = R$), which comprises all \equiv -degrees within $\deg_{\mathcal{I}}(R)$ that are greater than $\deg(R)$: this easily follows from Facts 2.10 and 2.12. On the other hand, if $S < R$ in $\deg_{\mathcal{I}}(R)$ then there is $k \geq 1$ such that $R \equiv S \oplus \text{Id}_k$, and all T such that $R \equiv T \oplus \text{Id}_k$ are $\equiv S$ (using Fact 2.5)(1), so for every $k \geq 1$ for which there is an S as above we can choose such a (unique up to \equiv) S and define $R_{-k} = S$. If there is no \leq -minimal element in $\deg_{\mathcal{I}}(R)$ then all ceers in $\deg_{\mathcal{I}}(R)$ which are smaller than R are \equiv to some R_{-k} , which form a chain $\cdots < R_{-k} < \cdots < R_{-1}$; thus the \mathcal{I} -degree $\deg_{\mathcal{I}}(R)$ consists of a \mathbb{Z} -chain of \equiv -degrees, namely the degrees of

$$\cdots < R_{-2} < R_{-1} < R < R_1 < R_2 < \cdots .$$

Definition 3.1. A ceer in this second type of minimal \mathcal{I} -degree we call a \mathbb{Z} -dark minimal ceer.

Example 3.2. Examples of dark minimal degrees are from Theorem 4.10 and Corollary 4.15 of [4]. As to examples of \mathbb{Z} -dark minimal ceers, they come from Theorem 4.11 and Corollary 4.15 of [4]: indeed, one can use the proof of Theorem 4.11, but without coding any ceer A , to build $\leq_{\mathcal{I}}$ -minimal dark ceers with finite classes. That the construction in Theorem 4.11 suffices is justified by Lemma 3.5 below.

The following fact was shown in [1], the right-to-left implication being already in [4, Lemmas 3.4 and 3.5]:

Fact 3.3. *R is a dark minimal ceer if and only if R has infinitely many classes and every c.e. set W which intersects infinitely many R -classes intersects every R -class.*

An easy consequence of this fact is the following:

Observation 3.4. *If R is a dark minimal ceer, then every pair of classes $[a]_R \neq [b]_R$ are computably inseparable. As a consequence, if R is a dark minimal ceer and $R \leq_{\mathcal{I}} S$ then $R \leq S$.*

Proof. Suppose that X is a computable set which separates $[a]_R$ and $[b]_R$. Either X or the complement of X must intersect infinitely many classes in R . But neither can intersect every class, since $[a]_R \subseteq X$ and $[b]_R \cap X = \emptyset$. This contradicts Fact 3.3. The latter claim follows from the fact that if R is a dark minimal ceer then no reduction $R \leq S \oplus \text{Id}_k$, for any $k \geq 1$, can hit Id_k since no R -equivalence class is computable. \square

In the next lemma we show the corresponding fact for the \mathbb{Z} -dark minimal ceers.

Lemma 3.5. *A ceer R is a \mathbb{Z} -dark minimal ceer if and only if it has infinitely many computable classes and every c.e. set W which intersects infinitely many R -classes intersects co-finitely many R -classes.*

Proof. Suppose that R is a \mathbb{Z} -dark minimal ceer. For every k , $R \equiv S \oplus \text{Id}_k$ for some S . By self-fullness of R and Observation 2.13, the reduction of R to $S \oplus \text{Id}_k$ is onto the classes of $S \oplus \text{Id}_k$, so

R has at least k computable classes. This is true for every k , so R has infinitely many computable classes. Suppose W is a c.e. set which intersects infinitely many R -classes. Since $R \upharpoonright W \leq R$, we see by \mathcal{I} -minimality of R that $R \upharpoonright W \oplus \text{Id}_k \equiv R$ for some $k \geq 0$, where again we let $R \upharpoonright W \oplus \text{Id}_0 = R \upharpoonright W$. Suppose now that W omits $\geq k + 1$ classes. Then from the inclusion $R \upharpoonright W \leq R$ and sending the k -classes of Id_k to k of the classes omitted by W it is possible to build a reduction $R \upharpoonright W \oplus \text{Id}_k \leq R$ which omits at least one class in its range. But this, together with $R \leq R \upharpoonright W \oplus \text{Id}_k$, contradicts Observation 2.13. It follows that W hits co-finitely many classes.

For the converse: First we check that, under the assumptions, R is dark. Suppose W enumerated an infinite transversal. Consider V any c.e. co-infinite subset of W . Since W is a transversal, V cannot contain co-finitely many classes in R . Yet V hits infinitely many classes in R , contradicting the hypothesis on R . Thus R is dark.

Since R has infinitely many computable classes, then for every k we can collapse any $k + 1$ of these together to find an E so that $E \oplus \text{Id}_k \equiv R$. Thus R is not in the \mathcal{I} -class of a dark minimal ceer, for choosing such an E and taking $R_{-k} = E$, we must have an infinite strictly descending chain provided by the \equiv -degrees of the various R_{-k} . Towards showing \mathcal{I} -minimality of R , suppose $X \leq_{\mathcal{I}} R$, where X is not finite. By Fact 2.7, $X \equiv_{\mathcal{I}} Y$ for some $Y \leq R$. By Remark 2.2, there is some W so that $Y \equiv R \upharpoonright W$. Since X is not finite, Y must not be finite, and thus W intersects infinitely many R -classes. Thus W intersects co-finitely many R -classes. Thus, by Fact 2.5(3), $Y \equiv R_{-m}$ for some m . Thus, $R \leq_{\mathcal{I}} R_{-m} \leq_{\mathcal{I}} X$. Thus R is \mathcal{I} -minimal. \square

We isolate the following fact which is immediate from Lemma 3.5:

Corollary 3.6. *If R is a \mathbb{Z} -dark minimal ceer, W is c.e., and W intersects infinitely many classes, then the R -closure of W is computable, and every class omitted from W is computable.*

Proof. The R -closure of W along with each of the finitely many omitted classes forms a partition of ω into finitely many c.e. sets. Thus, each of these sets is computable. \square

Remark 3.7. Notice that although in Fact 3.3 and Lemma 3.5 we distinguish between the cases of being dark minimal and being \mathbb{Z} -dark minimal for a ceer lying in a dark minimal \mathcal{I} -degree, it is correct to say, unifying the two cases, that if R lies in a dark minimal \mathcal{I} -degree then every c.e. set intersecting infinitely many R -equivalence classes intersects co-finitely many classes. If R is in the \mathcal{I} -degree of a dark minimal ceer, say $R \equiv R_0 \oplus \text{Id}_n$ where R_0 is dark minimal, then “co-finitely many classes” means in fact in this case “all classes except for the n computable classes”.

Definition 3.8. We say that \mathbf{a} is an \mathcal{I} -strongly minimal cover of the pair of \mathcal{I} -degrees \mathbf{d}, \mathbf{e} if \mathbf{d} and \mathbf{e} are \mathcal{I} -incomparable, $\mathbf{d}, \mathbf{e} <_{\mathcal{I}} \mathbf{a}$, and the only degrees $<_{\mathcal{I}} \mathbf{a}$ are $\leq_{\mathcal{I}} \mathbf{d}$ or $\leq_{\mathcal{I}} \mathbf{e}$.

Lemma 3.9. *If \mathbf{r}_1 and \mathbf{r}_2 are distinct dark minimal \mathcal{I} -degrees, then $\mathbf{r}_1 \oplus \mathbf{r}_2$ is an \mathcal{I} -strongly minimal cover of the pair $\mathbf{r}_1, \mathbf{r}_2$.*

Proof. Let R_1 and R_2 be in the \mathcal{I} -degrees $\mathbf{r}_1, \mathbf{r}_2$ respectively, and suppose $X <_{\mathcal{I}} R_1 \oplus R_2$. We want to show that either $X \leq_{\mathcal{I}} R_1$ or $X \leq_{\mathcal{I}} R_2$. We may assume that X is not finite, otherwise the claim is trivial. From Fact 2.7, we have that $X \equiv_{\mathcal{I}} X_1 \oplus X_2$ where $X_1 \leq R_1$, and $X_2 \leq R_2$. By the \mathcal{I} -minimality of R_1 , we have that either $R_1 \leq_{\mathcal{I}} X_1$ or X_1 is finite. Similarly, either $R_2 \leq_{\mathcal{I}} X_2$ or X_2 is finite. If both $R_1 \leq_{\mathcal{I}} X_1$ and $R_2 \leq_{\mathcal{I}} X_2$, then $R_1 \oplus R_2 \leq_{\mathcal{I}} X$ contradicting $X <_{\mathcal{I}} R_1 \oplus R_2$. Thus without loss of generality we may assume X_2 is finite. Then $X \equiv_{\mathcal{I}} X_1 \oplus X_2 \leq_{\mathcal{I}} X_1 \leq_{\mathcal{I}} R_1$. \square

Lemma 3.10. *If r_1 and r_2 are distinct dark minimal \mathcal{I} -degrees represented by R_1 and R_2 , x is even and y is odd, then the \mathcal{I} -degree of $(R_1 \oplus R_2)_{/(x,y)}$ is an \mathcal{I} -strongly minimal cover of the pair r_1 and r_2 .*

Proof. Suppose $X <_{\mathcal{I}} (R_1 \oplus R_2)_{/(x,y)}$: we want to show that either $X \leq_{\mathcal{I}} R_1$ or $X \leq_{\mathcal{I}} R_2$. We may assume that X is not finite, otherwise the claim is trivial. Take n so that $X \leq (R_1 \oplus R_2)_{/(x,y)} \oplus \text{Id}_n$. By Fact 2.3, we can write $X \equiv X_0 \oplus X_1$ where $X_0 \leq (R_1 \oplus R_2)_{/(x,y)}$ and $X_1 \leq \text{Id}_n$. Thus $X \equiv_{\mathcal{I}} X_0$, and since we are only considering X up to \mathcal{I} -degree, we may replace X by X_0 and thus we may assume that $X < (R_1 \oplus R_2)_{/(x,y)}$. Let $W = \text{range}(f)$ where f reduces $X \leq (R_1 \oplus R_2)_{/(x,y)}$. If $W \cap 2\omega$ hits infinitely many classes, then $\frac{W}{2}$ hits cofinitely many classes in R_1 . Similarly on the odd classes.

We first rule out the possibility that W hits both infinitely many even and infinitely many odd classes: if this were the case, then there would be h, k such that $\frac{W}{2}$ would omit h R_1 -classes and $\frac{W-1}{2}$ would omit k R_2 -classes. If f does not hit the $(R_1 \oplus R_2)_{/(x,y)}$ -class of x then f omits exactly $h+k-1$ classes, otherwise f omits exactly $h+k$ classes. Let j be the number of classes (either $h+k-1$ or $h+k$) omitted by f . Then by Fact 2.5(2), $X \oplus \text{Id}_j \equiv (R_1 \oplus R_2)_{/(x,y)}$, so $X \equiv_{\mathcal{I}} (R_1 \oplus R_2)_{/(x,y)}$. This contradicts the assumption that $X <_{\mathcal{I}} (R_1 \oplus R_2)_{/(x,y)}$.

Therefore, we may suppose that W hits only finitely many even classes, the case of W hitting only finitely many odd classes being similar. Now we aim to show that $X \leq_{\mathcal{I}} R_2$. Let k be the number of the finitely many even classes that are hit by f , and choose representatives a_0, \dots, a_{k-1} for these classes. Recall that $0, 1, \dots, k-1$ are representatives of the k equivalence classes of Id_k . Consider the function g computed by the following procedure: Given a number x , if $f(x)$ is even then search for the first a_i such that $f(x) (R_1 \oplus R_2)_{/(x,y)} a_i$ and let $g(x) = a_i$. If $f(x)$ is odd, then let $g(x) = f(x)$. Using g we see that if W does not hit the $(R_1 \oplus R_2)_{/(x,y)}$ -equivalence class of x then there is a reduction $X \leq \text{Id}_k \oplus R_2$, which, for each $i < k$, matches e.g. the $(R_1 \oplus R_2)_{/(x,y)}$ -equivalence class of a_i with the $\text{Id}_k \oplus R_2$ -equivalence class of $2i$. Otherwise there is a reduction $X \leq (\text{Id}_k \oplus R_2)_{/(0,y)} \equiv \text{Id}_{k-1} \oplus R_2$, where we assume without loss of generality that the $(R_1 \oplus R_2)_{/(x,y)}$ -equivalence classes of a_0 and x coincide. In either case, we have $X \leq_{\mathcal{I}} R_2$. \square

Lemma 3.11. *If R_1 and R_2 are \mathcal{I} -incomparable and are each dark minimal or \mathbb{Z} -dark minimal ceers, and $[x_0]_{R_1}$ and $[y_0]_{R_2}$ are non-computable, then, letting $x = 2x_0$ and $y = 2y_0 + 1$, we have that $R_1 \oplus R_2$ and $(R_1 \oplus R_2)_{/(x,y)}$ are \mathcal{I} -incomparable ceers. (Note that the condition that $[x]_{R_1}$ and $[y]_{R_2}$ are non-computable must hold if R_1 and R_2 are each dark minimal, by Observation 3.4.)*

Proof. Since any ceer $<_{\mathcal{I}}$ either $R_1 \oplus R_2$ or $(R_1 \oplus R_2)_{/(x,y)}$ must be $\leq_{\mathcal{I}}$ either R_1 or R_2 (from the previous two lemmas), we only need to show that $R_1 \oplus R_2$ and $(R_1 \oplus R_2)_{/(x,y)}$ are not \mathcal{I} -equivalent ceers, and thus it is enough to show that $(R_1 \oplus R_2)_{/(x,y)} \not\leq_{\mathcal{I}} R_1 \oplus R_2$. Suppose towards a contradiction that $(R_1 \oplus R_2)_{/(x,y)} \leq_{\mathcal{I}} R_1 \oplus R_2$. Then let n be so that $(R_1 \oplus R_2)_{/(x,y)} \leq R_1 \oplus R_2 \oplus \text{Id}_n$. Consider the composed reduction $R_1 \leq (R_1 \oplus R_2)_{/(x,y)} \leq R_1 \oplus R_2 \oplus \text{Id}_n$. In this reduction, let W be the set of elements sent into the copy of R_1 in $R_1 \oplus R_2 \oplus \text{Id}_n$. If W intersects only finitely many R_1 -classes, then $R_1 \leq_{\mathcal{I}} R_2$, which contradicts R_1 and R_2 being \mathcal{I} -incomparable. So, W intersects infinitely many R_1 -classes, thus it intersects co-finitely many R_1 -classes and misses only computable classes. Thus, x_0 is sent into R_1 in $R_1 \oplus R_2$. Similarly, y_0 is sent into R_2 in $R_1 \oplus R_2$ in the composed reduction $R_2 \leq (R_1 \oplus R_2)_{/(x,y)} \leq R_1 \oplus R_2 \oplus \text{Id}_n$. Thus the reduction $(R_1 \oplus R_2)_{/(x,y)} \leq R_1 \oplus R_2 \oplus \text{Id}_n$ sends x into R_1 and y into R_2 . Thus x and y are equivalent in $(R_1 \oplus R_2)_{/(x,y)}$ and their images are inequivalent in $R_1 \oplus R_2 \oplus \text{Id}_n$, which is a contradiction. \square

The previous Lemmas gives us two ways to build \mathcal{I} -strongly minimal covers of pairs of \mathcal{I} -incomparable minimal dark \mathcal{I} -degrees. This assumed that the degrees were represented by ceers which had a non-computable class. We now see that this is a necessary condition for constructing two incomparable \mathcal{I} -strongly minimal covers.

Lemma 3.12. *Let \mathbf{r} be the \mathcal{I} -degree of a \mathbb{Z} -dark minimal ceer R all of whose classes are computable. Let \mathbf{s} be an incomparable \mathcal{I} -degree. Then $\mathbf{r} \oplus \mathbf{s}$ is the join of \mathbf{r} and \mathbf{s} in the \mathcal{I} -degrees.*

Proof. Let S be any ceer. Let X be any ceer so that $R, S \leq_{\mathcal{I}} X$. We want to show that either $R \oplus S \leq_{\mathcal{I}} X$ or $R \leq_{\mathcal{I}} S$. Fix some n so that $R, S \leq X \oplus \text{Id}_n$ and let f and g give these reductions. Let W be the set of pairs $(2x, 2y + 1)$ so that $f(x)X \oplus \text{Id}_n g(y)$. Then $R \oplus S/W \leq X \oplus \text{Id}_n$. We have two cases to consider. Let $Z = \{x \mid \exists y(2x, 2y + 1) \in W\}$. Note that Z is R -closed.

Case 1: Z contains only finitely many R -classes. In this case, since each of these classes are computable, Z is computable. Let $R_0 = R \upharpoonright \omega \setminus Z$. Then $R \oplus S \equiv_{\mathcal{I}} R_0 \oplus S \leq X \oplus \text{Id}_n$, so we conclude $R \oplus S \leq_{\mathcal{I}} X$.

Case 2: Z contains infinitely many R -classes. In this case, we know that Z contains co-finitely many R -classes. Since each of the remaining classes are computable, Z is computable. Let $R_0 = R \upharpoonright Z$. Note that $R_0 \equiv_{\mathcal{I}} R$. We now give a map which reduces R_0 to S . For $x \in R_0$, search for some y so that $(2x, 2y + 1) \in W$. Send x to the first such y found. This gives a reduction of R_0 to S , thus $R \leq_{\mathcal{I}} S$. \square

Lemma 3.13. *If A is a \mathbb{Z} -dark minimal ceer and B is any ceer so that the pair A, B has two \mathcal{I} -incomparable \mathcal{I} -strongly minimal covers, then A has a non-computable class.*

Proof. Immediate from the preceding lemma. \square

4. CODING GRAPHS INTO THE PARTIAL ORDER $\mathbf{Dark}_{/\mathcal{I}}$ USING PARAMETERS

In the following by a *graph* we mean a structure $G = \langle V, E \rangle$ where V is a nonempty set of *vertices*, and E , called the *edge relation*, is an irreflexive and symmetric binary relation on V .

Definition 4.1. Given any dark \mathcal{I} -degree \mathbf{c} , we describe a graph $G_{\mathbf{c}}$ as follows:

- (vertices) the *vertex set* V of the graph are the \mathcal{I} -minimal degrees $\leq \mathbf{c}$.
- (edges) For each pair $\mathbf{d}, \mathbf{e} \in V$, we put an *edge* between \mathbf{d} and \mathbf{e} if and only if there are two incomparable \mathcal{I} -degrees $\mathbf{a}, \mathbf{b} \leq_{\mathcal{I}} \mathbf{c}$ which are both \mathcal{I} -strongly minimal covers of the pair \mathbf{d}, \mathbf{e} .

We will often say that we have an edge between ceers R and S if we have an edge between their \mathcal{I} -degrees.

We now show how to code any computable graph in $\mathbf{Dark}_{/\mathcal{I}}$, modulo some isolated vertices.

Theorem 4.2. *If $G = (V, E)$ is a computable graph, then there exists a dark degree \mathbf{c} so that $G_{\mathbf{c}}$ is isomorphic to the disjoint union of G with a graph which has no edges.*

Proof. Fix a computable presentation of G . We also fix a uniform c.e. sequence of pairwise $\leq_{\mathcal{I}}$ -incomparable dark minimal ceers $\{R_i \mid i \in \omega\}$: for this, just observe that by the proof of [4, Theorem 3.3] from any finite set R_0, \dots, R_n of dark minimal ceers we can uniformly find a dark minimal ceer R_{n+1} so that $R_{n+1} \not\leq R$ where $R = \bigoplus_{i \leq n} R_i$, and thus $R_{n+1} \not\leq_{\mathcal{I}} R$ by Observation 3.4;

at the same time, for each i , $R_i \not\leq_{\mathcal{I}} R_{n+1}$ otherwise (again by Observation 3.4) $R_i \leq R_{n+1}$, and thus $R_{n+1} \leq R_i \leq R$, by minimality of R_{n+1} . If R, E are ceers we say that the n -th column of E codes R (or R is copied in the n -th column of E) if for every x, y , $x R y$ if and only if $\langle n, x \rangle E \langle n, y \rangle$.

Requirements and strategies. We construct the ceer C (with \mathcal{I} -degree \mathbf{c}) with the following requirements:

- Code $_i$: Some column of C codes R_i .
- Dark $_j$: If W_j is infinite, then there are distinct $x, y \in W_j$ so that $x C y$.
- Edge $_{\langle i, j \rangle}$: If there is an edge between i and j in G , then some column of C codes $(R_i \oplus R_j)_{/(0,1)}$.

We have no requirement directly ensuring that there are no extra edges in $G_{\mathbf{c}}$. This will follow from the simple form of the ceer C that we construct.

The *priority order* of the requirements is

$$\text{Code}_0 < \text{Edge}_0 < \text{Dark}_0 < \dots < \text{Code}_j < \text{Edge}_j < \text{Dark}_j < \dots$$

A requirement Edge $_{\langle i, j \rangle}$ such that the graph G has an edge between i and j will be called *binding*. As we consider only symmetric irreflexive graphs, we assume that if $(i)_0 = (i)_1$ then Edge $_i$ is not binding, and Edge $_{\langle j, i \rangle}$ is not binding if $\langle i, j \rangle < \langle j, i \rangle$.

We outline the strategies to meet the requirements.

For the sake of the Code $_i$ -requirement, we act by picking a new column and determining that this column will copy R_i . It restrains this entire column.

For the sake of the Dark $_j$ -requirement, while the requirement is not satisfied (it becomes permanently satisfied when distinct numbers x, y appear such that $x, y \in W_j$ and $x C y$) we simply wait for W_j to enumerate two distinct numbers x, y which are not in columns restrained by higher priority requirements, then we collapse to a single class the entire columns of x and y .

We will use the following notations: $\omega^{[i]}$ denotes the i -th column of ω ; $\omega^{[i,j]} = \bigcup_{i \leq r \leq j} \omega^{[r]}$, and $\omega^{[i,j]} = \bigcup_{i \leq r < j} \omega^{[r]}$. It will follow from the construction that Dark $_j$ works with a parameter ϵ_j so that the columns restrained by higher priority requirements will be the columns $\omega^{[n]}$ for $n \leq \epsilon_j$; eventually ϵ_j stabilizes in the limit, and the interval $[0, \epsilon_j]$ is eventually partitioned into subintervals, each one being either a singleton $\{i\}$ so that in the i -th column codes a dark ceer (with R_0 coded in the 0-th column); or a subinterval $[a, b]$ so that all columns $\omega^{[i]}$ with $i \in [a, b]$ are collapsed to a single (clearly decidable) class. Being thus computably bijective with a uniform join of dark ceers and copies of Id_1 and being infinite, this finite set of columns can be viewed as a dark ceer R (see Fact 2.10). If an infinite W_j is contained in this finite set of columns, we need not do anything, as in this case W_j is not a transversal of C . Otherwise (as shown in the verification) from W_j one could find an infinite c.e. set which is a transversal contained in one column of C , which is impossible since each column of C is either finite or codes a dark ceer. In the following we will distinguish between *coding columns* in which we code dark ceers, and *column blocks*, comprised of finitely many consecutive columns of ω all collapsed to a single C -equivalence class which is decidable.

If Dark $_j$ acts by collapsing, it re-initializes all lower priority requirements, it leaves untouched all coding columns and column blocks in the restrained interval $[0, \epsilon_j]$, and collapses to a single class

all other columns up to the biggest column so far used in the construction: these newly collapsed columns contain also the witnesses x, y which are thus collapsed and Dark_j is permanently satisfied.

For the sake of the $\text{Edge}_{\langle i, j \rangle}$ -requirement, we act exactly as in the other Coding requirements Code_i . The only distinction is that the existence of any one of these Edge-requirements (i.e. whether or not it is binding) is determined by the computable graph G .

It follows that in the end C will consist of single coding columns (used for coding ceers of the form R_i or $(R_i \oplus R_j)_{/(0,1)}$) and column blocks comprised of finitely many consecutive columns of ω all collapsed to a single C -equivalence class. We say that a ceer with this structure is a *simple coding ceer*.

We first observe that the only dark minimal ceers $\leq_{\mathcal{I}} C$ are the R_i 's that we began with. To see this, let $E \leq_{\mathcal{I}} C$ be dark minimal. Firstly notice that $E \leq C$ by Observation 3.4, and E must reduce to a single coding column. In fact (again by Observation 3.4) no class can be mapped to a column block as no E -class is decidable, and no two distinct E -classes can be mapped to distinct coding columns by computable inseparability of E . So $E \leq R_i$, or $E \leq (R_i \oplus R_j)_{/(0,1)}$ for some i, j . In the former case E is equivalent to R_i , and in the latter case, Lemma 3.10 shows that E must be equivalent to either R_i or R_j .

Adequacy of the requirements. We now suppose that C is a simple coding ceer and has been constructed satisfying all the requirements and we verify that $G_{\mathcal{C}} \simeq G$ (where \simeq denotes isomorphism) modulo a set of isolated vertices, which are \mathcal{I} -degrees of \mathbb{Z} -dark minimal ceers with only computable classes.

We first observe that the only \mathbb{Z} -dark minimal ceers $\leq_{\mathcal{I}} C$ have only computable classes. Suppose towards a contradiction that A is a \mathbb{Z} -dark minimal ceer with a non-computable class $[m]_A$, and f is a reduction $f : A \leq C \oplus \text{Id}_r$ for some r which witnesses $A \leq_{\mathcal{I}} C$. Let $f(m) = 2 \cdot \langle n, x \rangle$, where clearly the n -th column is a coding column, say coding the ceer X , by undecidability of $[m]_A$. Let W be the computable set of y so that $f(y)$ does not land in the n -th column of C . By Corollary 3.6 W cannot hit infinitely many classes, as otherwise $[m]_A$, which is omitted by W , should be computable. Thus we must have that W intersects only finitely many classes. Thus the reduction f lands in the n -th column of C plus a finite collection of single equivalence classes of $C \oplus \text{Id}_r$. Thus $A \leq X \oplus \text{Id}_s$ for some s , so $A \leq_{\mathcal{I}} X$. Now, X is either a dark minimal ceer or one of $(R_i \oplus R_j)_{/(0,1)}$. Since a dark minimal ceer cannot bound a \mathbb{Z} -dark minimal ceer, the former case is impossible. In the latter case, we have that $A \equiv_{\mathcal{I}} (R_i \oplus R_j)_{/(0,1)}$ or $A \leq_{\mathcal{I}} R_i$ or $A \leq_{\mathcal{I}} R_j$. In any case A is not a \mathbb{Z} -dark minimal ceer, which is a contradiction.

Thus, the universe of $G_{\mathcal{C}}$ is comprised of the dark minimal ceers R_i along with perhaps some \mathbb{Z} -dark minimal ceers which have all computable classes. By Lemma 3.13, these are isolated points in $G_{\mathcal{C}}$.

If there is an edge between i and j in G , then we have $R_i \oplus R_j$ and $(R_i \oplus R_j)_{/(0,1)}$ being both $\leq C$. By Lemmas 3.9 and 3.10, we have an edge between R_i and R_j in $G_{\mathcal{C}}$.

Now, suppose that there is no edge between i and j in G . Then we do not place any columns in C of the form of $(R_i \oplus R_j)_{/(0,1)}$. Suppose X is an \mathcal{I} -strongly minimal cover below C of the pair R_i, R_j . Consider the composed reductions $R_i \leq_{\mathcal{I}} X \leq_{\mathcal{I}} C$ and $R_j \leq_{\mathcal{I}} X \leq_{\mathcal{I}} C$. By Observation 3.4, the first reduction in each of the two chains is \leq . By computable inseparability of the classes of R_i and R_j , these reductions reduce to single coding columns of C . Each coding column is either some R_k , or has the form $(R_k \oplus R_l)_{/(0,1)}$. In the latter case, by Lemma 3.10 only R_k and R_l reduce to that column. Thus, the two coding columns in which R_i and R_j are reducing to in C are different

columns. Thus we see $R_i \oplus R_j \leq X$, giving $R_i \oplus R_j \equiv_{\mathcal{I}} X$ as X is an \mathcal{I} -strongly minimal cover of the pair R_i, R_j . Thus we can only have one \mathcal{I} -strongly minimal cover of the pair R_i, R_j below C , and there is no edge between R_i and R_j in G_c .

The construction. In the formal construction we make use of several parameters: $\gamma_i(s)$, if defined, denotes the column in which at s we code R_i . $\epsilon_{\langle i,j \rangle}(s)$, if defined and $\text{Edge}_{\langle i,j \rangle}$ is binding, denotes the column in which at s we code $(R_i \oplus R_j)_{/(0,1)}$. The parameter $r(s)$ denotes the least number n so that the corresponding column is still *fresh* i.e. no parameter $\gamma_i(t)$ or $\epsilon_i(t)$ for $t \leq s$ was defined and $\geq n$. At each stage s there will always be a unique number i such that we define for the first time, or redefine, $\gamma_i(s)$. This will determine also the definition of $\epsilon_i(s)$ as

$$\epsilon_i(s) = \begin{cases} \gamma_i(s) + 1, & \text{if Edge}_i \text{ is binding,} \\ \gamma_i(s), & \text{otherwise.} \end{cases}$$

This means that we plan to code R_i in the $\gamma_i(s)$ -column. If there is an edge in the graph from $(i)_0$ to $(i)_1$ then we plan to code $(R_{(i)_0} \oplus R_{(i)_1})_{/(0,1)}$ in the next column, and otherwise Edge_i is not binding and does not need to be coded in any column. A requirement Dark_j is *satisfied* at s , if W_j has already enumerated a pair of distinct numbers x, y which C has already collapsed. Finally, at each stage s we define a ceer C_s , so that $C_0 \subseteq C_1 \subseteq \dots$, and the sequence $\{C_s \mid s \in \omega\}$ is c.e., so that $C = \bigcup_s C_s$ is our desired final ceer.

Stage 0. Let $\gamma_0(0) = 0$. Since $0 = \langle 0, 0 \rangle$, Edge_0 is not binding, thus $\epsilon_0(0) = 0$, and $r(0) = 1$. All other parameters are undefined. (The construction will ensure that γ_0 and ϵ_0 will never be initialized.) Let C_0 be the ceer generated by the c.e. set of pairs X where

$$X = \{\gamma_0(0)\} \times R_0 (= \{0\} \times R_0)$$

Here and below we use the notation: For $n \in \omega$ and E a ceer, $\{n\} \times E = \{(\langle n, x \rangle, \langle n, y \rangle) \mid (x, y) \in E\}$.

Stage $s + 1$. Let us say that Dark_j *requires attention* at $s + 1$ if $j \leq s$, $\epsilon_j = \epsilon_j(s)$ is defined, Dark_j is not as yet satisfied at the end of stage s , and there are distinct numbers $x, y \in W_{j,s+1} \cap \omega^{[\epsilon_j+1, r(s)]}$.

- (1) If some Dark_j requires attention, then pick the least such j and define $\gamma_{j+1}(s + 1) = r(s)$. This also determines $\epsilon_{j+1}(s + 1)$ (equal to $r(s) + 1$ or $r(s)$ depending on whether Edge_{j+1} is binding or not). Set to be undefined all γ_i and ϵ_i for all $i > j + 1$. Let C_{s+1} be the ceer generated by the c.e. set of pairs $C_s \cup X$ where, for the newly defined $\gamma_{j+1}, \epsilon_{j+1}$,

$$X = \begin{cases} (\omega^{[\epsilon_{j+1}, r(s)]})^2 \cup (\{\gamma_{j+1}\} \times R_{j+1}) \cup (\{\epsilon_{j+1}\} \times (R_{(j+1)_0} \oplus R_{(j+1)_1})_{/(0,1)}), & \text{if Edge}_{j+1} \\ & \text{is binding,} \\ (\omega^{[\epsilon_{j+1}, r(s)]})^2 \cup (\{\gamma_{j+1}\} \times R_{j+1}), & \text{otherwise,} \end{cases}$$

Notice that the pairs in $\omega^{[\epsilon_{j+1}, r(s)]}$ all C -collapse. Declare Dark_j *satisfied* (it will never become unsatisfied again), as Dark_j has C -collapsed two distinct numbers of W_j . Notice that Dark_j injures all lower priority Code- and Edge-requirements. The injured highest priority Code-requirement (namely, Code_{j+1}) starts anew on the fresh column $r(s)$, and, if binding, Edge_{j+1} starts anew on the next column.

- (2) If no Dark_j requires attention then let i be the least number such that $\gamma_i(s)$ is undefined. Define $\gamma_i(s + 1) = r(s)$. This determines $\epsilon_i(s + 1)$ and $r(s + 1)$ as well. Let C_{s+1} be the ceer generated by the c.e. set of pairs $C_s \cup X$ where, for the newly defined $\gamma_i = \gamma_i(s + 1)$

and $\epsilon_i = \epsilon_i(s + 1)$,

$$X = \begin{cases} C_s \cup (\{\gamma_i\} \times R_i) \cup (\{\epsilon_i\} \times (R_{(i)_0} \oplus R_{(i)_1})_{/(0,1)}), & \text{if Edge}_i \text{ is binding,} \\ C_s \cup (\{\gamma_i\} \times R_i), & \text{otherwise.} \end{cases}$$

Verification. We first observe that for every i , $\gamma_i = \lim_s \gamma_i(s)$ and $\epsilon_i = \lim_s \epsilon_i(s)$ exist. By the way we define ϵ_i we need only show that $\lim_s \gamma_i(s)$ exists, as $\epsilon_i = \gamma_i + 1$ if $\text{Edge}_{\langle(i)_0(i)_1\rangle}$ is binding, and $\epsilon_i = \gamma_i$ otherwise. This is seen by induction. The claim is trivial if $i = 0$ as $\gamma_0(s) = 0$ for every s .

Suppose that the claim is true of every $j \leq i$, and let s_0 be the least stage such that for every $s \geq s_0$ no such γ_j changes at s . Consider the requirement Dark_i . If it has already been satisfied by stage s_0 , or it will never act, then for every $s \geq s_0$ $\gamma_{i+1}(s) = \epsilon_i + 1$. On the other hand, if at some least stage $s_1 \geq s_0$ Dark_i acts, it becomes satisfied, by collapsing to a single class the column block $\omega^{[\epsilon_i+1, r(s_1)]}$. Then $\gamma_{i+1}(s) = r(s_1)$ for every $s \geq s_1$.

Finally we prove that C is dark, by showing that each Dark_i is satisfied. Suppose that W_i is infinite and let s_0 be a stage such that γ_i and ϵ_i never change after s_0 . Thus the columns $\omega^{[j]}$ with $j \leq \epsilon_i$ are partitioned in coding columns, and column blocks, and the intersection of C with these columns will never change after s_0 . Call E this intersection. Clearly there is a computable bijection f of $\omega^{[0, \epsilon_i]}$ onto ω under which E is translated into a ceer of the form $E' = E_1 \oplus \dots \oplus E_m$ where each E_k is \equiv to either some R_j or $(R_j \oplus R_h)_{/(0,1)}$ for some j, h , or a single class. Since the 0-th column codes R_0 and thus E' is not finite, it follows that E' is dark. By the same computable function f , $W_i \cap \omega^{[0, \epsilon_i]}$ is transformed into a c.e. set $W = f[W_i]$. If W_i were an infinite transversal of C and did not contain infinitely many elements in the complement of $\omega^{[0, \epsilon_i]}$, then W minus a finite set would be an infinite transversal of E' , which would contradict the darkness of E' . Thus if W_i is an infinite transversal of C , then there are distinct $x, y \in W_i$ such that $x, y \notin \omega^{[0, \epsilon_i]}$. But then Dark_i would act for the sake of some such pair x, y , and thus Dark_i would collapse such a pair x, y . We conclude that W_i is not a transversal of C . \square

Remark 4.3. Notice that instead of choosing 0, 1 when adding $(R_i \oplus R_j)_{/(0,1)}$ to C we could have chosen any pair (x, y) with x even and y odd, and added $(R_i \oplus R_j)_{/(x,y)}$. The argument in the previous proof relies on the fact that we can apply Lemma 3.11 which works as long as the equivalence classes of x and y are not computable. Since R_i and R_j are dark minimal, Observation 3.4 shows that all of their equivalence classes are non-computable.

Remark 4.4. In the proof of Theorem 4.2 we start with a uniform c.e. sequence of $\leq_{\mathcal{I}}$ -incomparable dark minimal ceers $\{R_i \mid i \in \omega\}$, as we are tacitly assuming that we need to code a graph with infinitely many vertices. If we need to code a finite graph, we can simply code a finite sequence of incomparable dark minimal ceers R_i along with ceers $(R_i \oplus R_j)_{/(0,1)}$ to code edges between i and j . We needn't even have any Dark_i requirements. In this case the \mathcal{I} -degree \mathbf{c} is the \mathcal{I} -degree of the uniform join of finitely many dark ceers, and thus it is automatically dark.

5. INTERPRETING $(\mathbb{N}, +, \times)$ IN THE PARTIAL ORDER $\mathbf{Dark}_{/\mathcal{I}}$ WITHOUT PARAMETERS

We are now ready to show how to give a definition of $(\mathbb{N}, +, \times)$ without parameters. As described in section 1.1, our strategy is to consider the codes \mathbf{c} so that $G_{\mathbf{c}}$ encodes a model of Robinson's system Q . Then the key is in encoding all finite functions to define an equivalence relation which picks out which elements of these different models of Q are the same number. By encoding all finite

functions, we see that this equivalence relation is at least correct on all standard numbers. Finally, our copy of \mathbb{N} will be the equivalence classes of the standard natural numbers, which we will isolate as the classes that have a representative in every encoded model of Q .

Definition 5.1. Let $\mathcal{P} = (P, \leq)$ be a poset, and $n \geq 1$. An relation $R \subseteq P^n$ is said to be \emptyset -definable in \mathcal{P} if there is a first order formula without parameters $\varphi(\vec{x})$ in the language of posets (with \vec{x} an n -tuple of variables, and all free variables of φ are in \vec{x}) such that, for every $\vec{a} \in P^n$,

$$R(\vec{a}) \Leftrightarrow \mathcal{P} \models \varphi(\vec{a}).$$

Corollary 5.2. *There are \emptyset -definable relations $V(x, c), E(x, y, c), NI(x, c)$ on $\mathbf{Dark}_{/\mathcal{I}}$ such that if c is a dark \mathcal{I} -degree then*

- $G_c = \{\mathbf{x} \mid V(\mathbf{x}, c)\}$,
- $E(\mathbf{x}, \mathbf{y}, c)$ if and only if $\mathbf{x}, \mathbf{y} \in G_c$ and there is an edge between \mathbf{x} and \mathbf{y} ,
- $\{\mathbf{x} \mid NI(\mathbf{x}, c)\}$ is the set on non-isolated vertices of G_c .

Proof. Immediate as the definitions of vertices and edges in Definition 4.1 are given in terms of minimality and existence of strong minimal covers for pairs, which are first order properties in the language of posets. \square

Remark 5.3. From the previous corollary, we see how to effectively translate any sentence σ in the language of graph theory (just the binary edge relation) into a formula $\tilde{\sigma}(w)$ of posets with free variable w such that for every dark \mathcal{I} -degree c ,

$$G_c \models \sigma \Leftrightarrow \mathbf{Dark}_{/\mathcal{I}} \models \tilde{\sigma}(c).$$

We will refer to the following result:

Theorem 5.4. *There is a computable graph G without isolated vertices in which $(\mathbb{N}, +, \times)$ is first order \emptyset -definable, that is there are first-order formulas without parameters $U, \varphi_+, \varphi_\times$ in the language of graphs defining respectively the subset which is the universe of the copy of \mathbb{N} and the operations $+, \times$ in G .*

Proof. See item 1(c)14 of the list of def-complete structures of [19], or see [18, Theorem 5.5.1]. \square

Remark 5.5. In view of Theorem 5.4 in coding G in $\mathbf{Dark}_{/\mathcal{I}}$ we only need the subset of the vertices of G_c which is comprised of the non-isolated vertices (all of them being dark minimal). Since by Corollary 5.2 this subgraph is recognizable in a first order way from parameter c , henceforth we shall use the symbol G_c to denote this subgraph, so that G_c is henceforth understood to be without isolated vertices and $G \simeq G_c$.

Fix a graph G as in Theorem 5.4. So there are first-order formulas $U, \varphi_+, \varphi_\times$ and a mapping $\sigma \mapsto \sigma^\circ$ from arithmetical formulas to formulas in the language of graphs so that the following hold, where $:=$ denotes syntactic equality: $(+(x, y, z))^\circ := \varphi_+(x, y, z)$, $(\times(x, y, z))^\circ := \varphi_\times(x, y, z)$, the mapping $^\circ$ commutes (modulo $:=$) with propositional connectives, $(\forall x \sigma)^\circ := (\forall x)(U(x) \rightarrow \sigma^\circ)$, $(\exists x \sigma)^\circ := (\exists x)(U(x) \wedge \sigma^\circ)$, and finally for every sentence σ , $\mathbb{N} \models \sigma$ if and only if $G \models \sigma^\circ$.

Remark 5.6. Using Corollary 5.2 we see that the following binary relation $U^c(x)$, and quaternary relations $\varphi_+^c(x, y, z), \varphi_\times^c(x, y, z)$ (corresponding to $U(x), \varphi_+(x, y, z), \varphi_\times(x, y, z)$ mentioned above) are \emptyset -definable in $\mathbf{Dark}_{/\mathcal{I}}$:

- $U^c(\mathbf{x})$ if and only if $G_c \models U(\mathbf{x})$ (henceforth let $U^c = \{\mathbf{x} \mid U^c(\mathbf{x})\}$),

- $\varphi_+^c(\mathbf{x}, \mathbf{y}, \mathbf{z})$ if and only if $G_c \models \varphi_+(\mathbf{x}, \mathbf{y}, \mathbf{z})$,
- $\varphi_\times^c(\mathbf{x}, \mathbf{y}, \mathbf{z})$ if and only if $G_c \models \varphi_\times(\mathbf{x}, \mathbf{y}, \mathbf{z})$.

For every $\mathbf{c} \in \mathbf{Dark}_{/\mathcal{I}}$ we can regard the triple $(U^c, \varphi_+^c, \varphi_\times^c)$ as a structure for the arithmetical language $+, \times$. From these, we also have \emptyset -definable relations $\varphi_S^c(x, y), \leq^c(x, y), \mathbf{0}^c(x)$ in $\mathbf{Dark}_{/\mathcal{I}}$, corresponding to the formulas defining in G the successor operation, the natural ordering on \mathbb{N} , and the number 0, respectively.

Definition 5.7. A $\mathbf{c} \in \mathbf{Dark}_{/\mathcal{I}}$ is a *good code* if, in G_c , $(U^c, \varphi_+^c, \varphi_\times^c)$ gives a model of Robinson's system Q .

Corollary 5.8. *The set of good codes is \emptyset -definable in the $\mathbf{Dark}_{/\mathcal{I}}$ degrees.*

Proof. This immediately follows from the fact that Q is finitely axiomatizable. \square

Remark 5.9. If \mathbf{c} is a good code then $\varphi_S^c, \varphi_+^c, \varphi_\times^c$ define operations in U^c , and $\mathbf{0}^c$ is a distinguished element of U^c .

We now begin encoding functions between our different models U^c and $U^{c'}$. The goal of these functions is to def

Definition 5.10. For any pair of dark minimal \mathcal{I} -degrees (\mathbf{x}, \mathbf{y}) , we say that a graph of the form $\mathbf{x} E \mathbf{a} E \mathbf{d} E \mathbf{y}$ and $\mathbf{a} E \mathbf{b} E \mathbf{c} E \mathbf{a}$ (where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are distinct, and distinct from \mathbf{x}, \mathbf{y}) is a *graph-label for the pair (\mathbf{x}, \mathbf{y})* . See Figure 1.

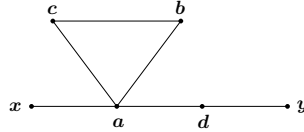


FIGURE 1. A graph-label for the pair (\mathbf{x}, \mathbf{y})

Definition 5.11. Given a set $F = \{(\mathbf{x}_i, \mathbf{y}_i) \mid i < n\}$ of pairs of dark minimal \mathcal{I} -degrees, we say that an $\mathbf{f} \in \mathbf{Dark}_{/\mathcal{I}}$ is a *name for F* if $G_{\mathbf{f}}$ is comprised of the union of graph-labels for the pairs $(\mathbf{x}_i, \mathbf{y}_i)$, where the various quadruples $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are distinct for each pair, and not appearing in F .

Lemma 5.12. *If F is a finite set of pairs of ceers, each of which is either dark minimal or is \mathbb{Z} -dark minimal with a non-computable class, then there is a dark \mathcal{I} -degree which is a name for the set of pairs of \mathcal{I} -degrees corresponding to the pairs of ceers in F .*

Proof. Fix a pair (X, Y) in F . Without loss of generality, we assume that $[0]_X$ and $[0]_Y$ are non-computable. Let A, B, C, D be distinct dark minimal ceers with \mathcal{I} -degrees not mentioned in F , and chosen only for this pair. Then we construct

$$\begin{aligned} Z_{(X,Y)} = & X \oplus A \oplus D \oplus Y \oplus B \oplus C \\ & \oplus (X \oplus A)_{/(0,1)} \oplus ((A \oplus D)_{/(0,1)}) \oplus ((D \oplus Y)_{/(0,1)}) \oplus \\ & ((A \oplus B)_{/(0,1)}) \oplus ((B \oplus C)_{/(0,1)}) \oplus (A \oplus C)_{/(0,1)}. \end{aligned}$$

Now that we have constructed $Z_{(X,Y)}$ for each pair $(X, Y) \in F$, define

$$f = \bigoplus_{(X,Y) \in F} Z_{(X,Y)}.$$

Since there are only finitely many direct summands in f , and each of these are dark, f is dark by Fact 2.10 (2). Let \mathbf{f} denote the \mathcal{I} -degree of f .

First we check that we have no unwanted vertices, i.e. the only minimal \mathcal{I} -degrees below \mathbf{f} are equal to the \mathcal{I} -degrees of the ceers X, Y, A, B, C, D that we placed there. If R has minimal \mathcal{I} -degree and $R \leq_{\mathcal{I}} f$, then by Fact 2.7 $R \equiv_{\mathcal{I}} \bigoplus_j E_j$ where each E_j is \leq one of the summands in the definition of f . That is, each E_j is either $\leq_{\mathcal{I}}$ some dark minimal ceer or E_j is $\leq_{\mathcal{I}} (S \oplus T)_{/(0,1)}$ where S and T are in dark minimal \mathcal{I} -degrees. Thus by Lemma 3.10, each E_j is either finite or its \mathcal{I} -degree is $\geq_{\mathcal{I}}$ a dark minimal \mathcal{I} -degree, namely the \mathcal{I} -degree of one of the summands. Thus since R is not finite, the \mathcal{I} -degree of one of the E_j is $\geq_{\mathcal{I}}$ the dark minimal \mathcal{I} -degree of one of the summands, and since R has minimal \mathcal{I} -degree, R is \mathcal{I} -equivalent to one of the summands.

Next we check that f codes exactly the edges we intended. Since the equivalence class of 0 is non-computable in all of the ceers X, Y, A, B, C, D that we consider, if we place columns for X, A , and $(X \oplus A)_{/(0,1)}$, we have ensured that the \mathcal{I} -degrees of X and A have two \mathcal{I} -incomparable \mathcal{I} -strongly minimal covers below \mathbf{f} . Similarly for the pairs (A, D) , (D, Y) , (A, B) , (B, C) , and (A, C) . Thus by Lemmas 3.9 and 3.10, f successfully codes every edge that we intended. Suppose now the \mathcal{I} -degree of X is an \mathcal{I} -strongly minimal cover below \mathbf{f} of the \mathcal{I} -degrees of the pair R_1 and R_2 , which are minimal \mathcal{I} -degrees below \mathbf{f} between which we did not explicitly code an edge. We may assume that R_1 and R_2 are among the summands we used to create f , in particular the equivalence class of 0 is non-computable. Then consider the pair of reductions $R_1 \leq f \oplus \text{Id}_n$ and $R_2 \leq f \oplus \text{Id}_n$, which we get by composing the reductions $R_1 \leq X \oplus \text{Id}_k \leq f \oplus \text{Id}_n$ and $R_2 \leq X \oplus \text{Id}_k \leq f \oplus \text{Id}_n$, respectively. Since the R_1 -equivalence class of 0 is not computable, its image under the reduction $R_1 \leq f \oplus \text{Id}_n$ must be in some column of f (not in Id_n). Let W be the set of elements whose image is not in the same column under the reduction $R_1 \leq f \oplus \text{Id}_n$. Since $[0]_{R_1}$ is not computable and it is not intersected by W , we must have that W contains only finitely many R_1 -classes by Remark 3.7. Thus $R_1 \leq_{\mathcal{I}}$ this one column of f . Similarly for R_2 . Since we did not explicitly code an edge between R_1 and R_2 , there is no column of f which is $\geq_{\mathcal{I}} R_1$ and $\geq_{\mathcal{I}} R_2$. So we see that $R_1 \oplus R_2 \leq_{\mathcal{I}} X$. Thus, since the \mathcal{I} -degree of X is assumed to be an \mathcal{I} -strongly minimal cover of the \mathcal{I} -degrees of the pair R_1, R_2 , we have $X \equiv_{\mathcal{I}} R_1 \oplus R_2$. Thus there can only be one \mathcal{I} -strongly minimal cover $\leq_{\mathcal{I}} \mathbf{f}$ of any pair of minimal \mathcal{I} -degrees aside from the pairs where we intended to place an edge. Thus there are no unwanted edges.

Therefore the \mathcal{I} -degree \mathbf{f} of f is the desired name for F . □

For \mathcal{I} -degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ define

$$[\mathbf{a}, \mathbf{b}]^{U^c} = \{\mathbf{x} \in U^c \mid \mathbf{a} \leq^c \mathbf{x} \leq^c \mathbf{b}\}.$$

This is clearly a ternary relation in $\mathbf{a}, \mathbf{b}, \mathbf{c}$, which is \emptyset -definable in $\mathbf{Dark}_{/\mathcal{I}}$.

Definition 5.13. On pairs of dark \mathcal{I} -degrees we define the equivalence relation $(\mathbf{c}, \mathbf{d}) \sim (\mathbf{c}', \mathbf{d}')$ if the two pairs coincide, or \mathbf{c} and \mathbf{c}' are good codes, $\mathbf{d} \in U^c$ and $\mathbf{d}' \in U^{c'}$, and there exists a name \mathbf{f} for a set of pairs F which is an order-preserving bijection between $[\mathbf{0}^c, \mathbf{d}]^{U^c}$ and $[\mathbf{0}^{c'}, \mathbf{d}']^{U^{c'}}$. That is, $G_{\mathbf{f}}$ is a graph so that for every $\mathbf{x} \in [\mathbf{0}^c, \mathbf{d}]^{U^c}$, there exists a unique \mathcal{I} -minimal \mathbf{y} so that there is a graph-label for the pair (\mathbf{x}, \mathbf{y}) contained in $G_{\mathbf{f}}$. Further, this \mathbf{y} is always in $[\mathbf{0}^{c'}, \mathbf{d}']^{U^{c'}}$. Furthermore, this map that sends $\mathbf{x} \in [\mathbf{0}^c, \mathbf{d}]^{U^c}$ to this unique \mathbf{y} is an order-preserving bijection between $[\mathbf{0}^c, \mathbf{d}]^{U^c}$ and $[\mathbf{0}^{c'}, \mathbf{d}']^{U^{c'}}$.

Lemma 5.14. *The relation \sim is \emptyset -definable in the collection of dark \mathcal{I} -degrees.*

Proof. This is straightforward from the definition of \sim and Remark 5.6. \square

Finally, we define:

Definition 5.15. Let \mathcal{N} be the set of \sim -equivalence classes of pairs (\mathbf{c}, \mathbf{d}) of dark \mathcal{I} -degrees so that for every good code \mathbf{c}' , there exists a \mathbf{d}' so that $(\mathbf{c}, \mathbf{d}) \sim (\mathbf{c}', \mathbf{d}')$.

Lemma 5.16. \mathcal{N} is \emptyset -definable in the dark \mathcal{I} -degrees.

Proof. By Corollary 5.8 and Lemma 5.14. \square

Lemma 5.17. Let \mathbf{c} be a good code so that $(U^{\mathbf{c}}, \varphi_+^{\mathbf{c}}, \varphi_{\times}^{\mathbf{c}}) \simeq (\mathbb{N}, +, \times)$. Then $\mathcal{N} = \{[(\mathbf{c}, \mathbf{d})]_{\sim} \mid \mathbf{d} \in U^{\mathbf{c}}\}$ (where $[(\mathbf{c}, \mathbf{d})]_{\sim}$ denotes the equivalence class of the pair (\mathbf{c}, \mathbf{d}) under \sim).

Proof. Every model of Robinson's Q has a standard part isomorphic to $(\mathbb{N}, +, \cdot)$. Note that if R is in a degree in $U^{\mathbf{c}'}$ for any good \mathbf{c}' , then R must have a non-computable class by Lemma 3.13. By Lemma 5.12, this shows that $\{[(\mathbf{c}, \mathbf{d})]_{\sim} \mid \mathbf{d} \in U^{\mathbf{c}}\} \subseteq \mathcal{N}$. For the converse, if $[(\mathbf{c}', \mathbf{d}')]_{\sim} \in \mathcal{N}$, then by definition of \mathcal{N} , there must be some $\mathbf{d} \in U^{\mathbf{c}}$ so that $(\mathbf{c}', \mathbf{d}') \sim (\mathbf{c}, \mathbf{d})$. Thus $\mathcal{N} \subseteq \{[(\mathbf{c}, \mathbf{d})]_{\sim} \mid \mathbf{d} \in U^{\mathbf{c}}\}$. \square

This allows us to define, without parameters, $+$ and \times on \mathcal{N} . For instance,

Definition 5.18. Let \mathbf{c} be a good code so that $(U^{\mathbf{c}}, \varphi_+^{\mathbf{c}}, \varphi_{\times}^{\mathbf{c}}) \simeq (\mathbb{N}, +, \times)$. We define $[(\mathbf{c}, \mathbf{d})]_{\sim} + [(\mathbf{c}', \mathbf{d}')]_{\sim} = [(\mathbf{c}, \varphi_+^{\mathbf{c}}(\mathbf{d}, \mathbf{d}'))]_{\sim}$, and similarly for the other operation.

Lemma 5.19. The definition of $+$ and \times on \mathcal{N} does not depend on the choice of \mathbf{c} and is \emptyset -definable in the partial order of $\mathbf{Dark}_{/\mathcal{I}}$ degrees.

Proof. The first claim is immediate from the definitions. We can define addition by saying that $[(\mathbf{c}, \mathbf{d})]_{\sim} + [(\mathbf{c}', \mathbf{d}')]_{\sim} = [(\mathbf{c}'', \mathbf{d}'')]_{\sim}$ if and only if there exists a good code $\hat{\mathbf{c}}$ and $\mathbf{e}, \mathbf{e}', \mathbf{e}'' \in U^{\hat{\mathbf{c}}}$ so that $(\mathbf{c}, \mathbf{d}) \sim (\hat{\mathbf{c}}, \mathbf{e})$, $(\mathbf{c}', \mathbf{d}') \sim (\hat{\mathbf{c}}, \mathbf{e}')$, $(\mathbf{c}'', \mathbf{d}'') \sim (\hat{\mathbf{c}}, \mathbf{e}'')$, and that $\varphi_+^{\hat{\mathbf{c}}}(\mathbf{e}, \mathbf{e}') = \mathbf{e}''$. \square

It is immediate that $(\mathcal{N}, +, \times)$ is isomorphic to $(\mathbb{N}, +, \times)$, and thus we have proved:

Theorem 5.20. There is an interpretation without parameters of $(\mathbb{N}, +, \times)$ in the partial order $\mathbf{Dark}_{/\mathcal{I}}$. Thus $\text{Th}(\mathbf{Dark}_{/\mathcal{I}})$ is computably isomorphic to $\text{Th}^1(\mathbb{N})$.

Proof. We have just shown that $\text{Th}^1(\mathbb{N}) \leq_1 \text{Th}(\mathbf{Dark}_{/\mathcal{I}})$. On the other hand, $\text{Th}(\mathbf{Dark}_{/\mathcal{I}}) \leq_1 \text{Th}^1(\mathbb{N})$ by Lemma 2.9. \square

Corollary 5.21. $\text{Th}(\mathbf{Dark})$ and $\text{Th}(\mathbf{Ceers})$ are computably isomorphic to $\text{Th}^1(\mathbb{N})$.

Proof. \mathcal{I} -equivalence is \emptyset -definable in \mathbf{Dark} by Lemma 2.8 and \mathbf{Dark} is \emptyset -definable in \mathbf{Ceers} by [4, Corollary 8.1]. Thus we have $\text{Th}^1(\mathbb{N}) \leq_1 \text{Th}(\mathbf{Dark}_{/\mathcal{I}}) \leq_1 \text{Th}(\mathbf{Dark}) \leq_1 \text{Th}(\mathbf{Ceers}) \leq_1 \text{Th}^1(\mathbb{N})$. \square

Finally,

Theorem 5.22. The theory of the partial order of $\mathbf{Ceers}_{/\mathcal{I}}$ is 1-equivalent to the theory of true arithmetic.

Proof. We use the same definition of G_c to code graphs and the same definition of \sim to give the same definition of $(\mathbb{N}, +, \cdot)$. The only subtlety in this case is that perhaps \mathbf{Id} , the \mathcal{I} -degree of Id , could be in G_c for some c . Lemma 5.12 does not give us names for sets of pairs including \mathbf{Id} . This is solved just as the case of \mathbb{Z} -dark minimal ceers with all computable classes. Namely, such elements must be isolated if they appear in G_c .

By [4, Observation 5.1 and Lemma 6.5], \mathbf{Id} has a least upper bound with every \mathcal{I} -degree. Thus, it cannot have two \mathcal{I} -strongly minimal covers with any degree and thus if it appears in G_c , it is isolated, and thus makes no difference in our definition of $(\mathbb{N}, +, \cdot)$. \square

6. CODING GRAPHS INTO THE PARTIAL ORDER $\mathbf{Light}_{/\mathcal{I}}$ USING PARAMETERS

We now turn our attention to the structure $\mathbf{Light}_{/\mathcal{I}}$. We once again need to show that any computable graph can be encoded in $\mathbf{Light}_{/\mathcal{I}}$. To do this, we use the analogous result in $\mathbf{Dark}_{/\mathcal{I}}$ and an embedding ι of $\mathbf{Dark}_{/\mathcal{I}}$ into $\mathbf{Light}_{/\mathcal{I}}$.

Recall that the symbol \mathbf{Id} denotes the \mathcal{I} -degree of the identity ceer Id .

Definition 6.1. We say that a light \mathcal{I} -degree e is *light minimal* if it is $>_{\mathcal{I}} \mathbf{Id}$ and $(\mathbf{Id}, e)_{\mathcal{I}}$ is empty, where $(\mathbf{Id}, e)_{\mathcal{I}}$ is the interval of \mathcal{I} -degrees x such that $\mathbf{Id} <_{\mathcal{I}} x <_{\mathcal{I}} e$.

Note that the property of being light minimal is \emptyset -definable in the partial order $\mathbf{Light}_{/\mathcal{I}}$. The following is Theorem 6.2 in [4].

Lemma 6.2. *The map $\iota : X \mapsto X \oplus \text{Id}$ induces an embedding of $\mathbf{Dark}_{/\mathcal{I}}$ into $\mathbf{Light}_{/\mathcal{I}}$.*

Proof. For two dark ceers X, Y we have

$$\begin{aligned} X \oplus \text{Id} \leq_{\mathcal{I}} Y \oplus \text{Id} &\Leftrightarrow X \oplus \text{Id} \leq Y \oplus \text{Id} \\ &\Leftrightarrow X \leq Y \oplus \text{Id} \end{aligned}$$

the last equivalence coming from that fact that $Y \oplus \text{Id} \oplus \text{Id} \equiv Y \oplus \text{Id}$ and thus if $X \leq Y \oplus \text{Id}$ then $X \oplus \text{Id} \leq Y \oplus \text{Id}$. On the other hand, $X \leq Y \oplus \text{Id}$ if and only if $X \leq Y \oplus \text{Id}_k$ for some k , because the darkness of X guarantees that the reduction cannot be infinite into the Id part. But since $X \leq Y \oplus \text{Id}_k$, for some k , is just the definition of $X \leq_{\mathcal{I}} Y$, we see

$$X \leq_{\mathcal{I}} Y \Leftrightarrow X \oplus \text{Id} \leq_{\mathcal{I}} Y \oplus \text{Id}.$$

\square

We will thus also use ι to refer to the induced map on \mathcal{I} -degrees.

Lemma 6.3. *If X is dark, Y is light, and $Y \leq X \oplus \text{Id}$, then $Y \equiv \text{Id}$ or there is a dark ceer $Z \leq X$ so that $Y \equiv Z \oplus \text{Id}$.*

Proof. Assume X, Y as in the statement of the lemma. Since $Y \leq X \oplus \text{Id}$, by Fact 2.3 we have $Y \equiv Z \oplus Y_0$ where $Z \leq X$ (and thus Z is either finite or dark) and $Y_0 \leq \text{Id}$. If Y_0 has only finitely many classes, then $Y \equiv Z \oplus \text{Id}_k$, but then Z is dark and so is its uniform join with a finite ceer, which is impossible since Y is light. It follows that $Y_0 \equiv \text{Id}$ and thus if Z is finite then $Y \equiv \text{Id}$, otherwise $Y \equiv Z \oplus \text{Id}$ for some dark Z . \square

Thus, the ceers of the form $X \oplus \text{Id}$ with X dark form an initial segment in the light ceers $> \text{Id}$.

Theorem 6.4. *Let C be dark and let \mathbf{c} denote its \mathcal{I} -degree. Then $\mathbf{Dark}_{/\mathcal{I}}(\leq_{\mathcal{I}} \mathbf{c}) \simeq (\mathbf{Id}, \iota(\mathbf{c}))_{\mathcal{I}}$.*

Proof. Immediate from Lemma 6.2 (which shows that ι is an order-theoretic embedding) and Lemma 6.3 which gives onto-ness. \square

Definition 6.5. For any $\mathbf{Light}_{/\mathcal{I}}$ degree \mathbf{c} , we associate a graph $H_{\mathbf{c}}$ as follows:

- (vertices) the *vertices* are the light minimal degrees $\leq_{\mathcal{I}} \mathbf{c}$;
- (edges) we place an *edge* between vertices \mathbf{d} and \mathbf{e} if and only if there are two incomparable \mathcal{I} -degrees $\mathbf{a}, \mathbf{b} \leq_{\mathcal{I}} \mathbf{c}$ which are a light \mathcal{I} -strongly minimal cover of the pair \mathbf{d} and \mathbf{e} , i.e. $\mathbf{d}, \mathbf{e} \leq_{\mathcal{I}} \mathbf{a}, \mathbf{b}$ and the only light \mathcal{I} -degrees $<_{\mathcal{I}} \mathbf{a}$ or $<_{\mathcal{I}} \mathbf{b}$ are \mathbf{d}, \mathbf{e} and \mathbf{Id} .

Lemma 6.6. *For any dark ceer \mathbf{c} , $H_{\iota(\mathbf{c})} \simeq G_{\mathbf{c}}$.*

Proof. This is immediate from Theorem 6.4: For a dark \mathcal{I} -degree \mathbf{x} , \mathbf{x} is \mathcal{I} -minimal if and only if $\iota(\mathbf{x})$ is light-minimal. Further, for dark degrees $\mathbf{x}, \mathbf{y}, \mathbf{z}$, \mathbf{z} is an \mathcal{I} -strongly minimal cover of \mathbf{x}, \mathbf{y} if and only if $\iota(\mathbf{z})$ is a light \mathcal{I} -strongly minimal cover of $\iota(\mathbf{x}), \iota(\mathbf{y})$. \square

Corollary 6.7. *For any computable graph G , there is a light \mathcal{I} -degree \mathbf{c} so that $H_{\mathbf{c}}$ is isomorphic to the disjoint union of G with a graph which has no edges.*

Proof. By Theorem 4.2 and Lemma 6.6. \square

7. INTERPRETING $(\mathbb{N}, +, \times)$ IN THE PARTIAL ORDER $\mathbf{Light}_{/\mathcal{I}}$ WITHOUT PARAMETERS

Following what we did for dark degrees in Definition 5.7 (and using the notations therein exploited), we define a light \mathcal{I} -degree \mathbf{c} to be *good* if in $H_{\mathbf{c}}$, the triple $(U^{\mathbf{c}}, \varphi_+^{\mathbf{c}}, \varphi_{\times}^{\mathbf{c}})$ gives a model of Robinson's Q . Recall that ι is an initial embedding of $\mathbf{Dark}_{/\mathcal{I}}$ into $\mathbf{Light}_{/\mathcal{I}} \setminus \{\mathbf{Id}\}$. So, we have a good code \mathbf{c} so that $H_{\mathbf{c}}$ codes $(\mathbb{N}, +, \cdot)$. Our goal is to define \mathcal{N} as in Section 5, but we have to use a different coding for finite functions. The reason is that we may have good codes which are $\mathbf{Light}_{/\mathcal{I}}$ -degrees which are not in the image of ι (the embedding introduced in Lemma 6.2), thus we cannot use the ι -image of the construction for names in $\mathbf{Dark}_{/\mathcal{I}}$.

Throughout the section an \mathcal{I} -strongly minimal cover of an \mathcal{I} degree \mathbf{x} means an \mathcal{I} degree $\mathbf{y} >_{\mathcal{I}} \mathbf{x}$ such that the interval $[\mathbf{Id}, \mathbf{y}]_{\mathcal{I}}$ is exactly the interval $[\mathbf{Id}, \mathbf{x}]_{\mathcal{I}}$. Since we will consider only \mathcal{I} -strongly minimal covers of light \mathcal{I} -degrees, they will be light as well.

Definition 7.1. Let $F = \{\{\mathbf{a}_i, \mathbf{b}_i\} \mid i \in S\}$ be a set of unordered pairs of light minimal \mathcal{I} -degrees so that the \mathbf{a}_i 's and the \mathbf{b}_j 's are distinct (including $\mathbf{a}_i \neq \mathbf{b}_j$ for any $i, j \in S$). We say that a light \mathcal{I} -degree \mathbf{f} is a *name for F* if the only light minimal \mathcal{I} -degrees below \mathbf{f} are $\{\mathbf{a}_i, \mathbf{b}_i \mid i \in S\}$, and the $\{\mathbf{a}_i, \mathbf{b}_i\}$'s are the only pairs $\{\mathbf{c}, \mathbf{d}\}$ of light minimal \mathcal{I} -degrees less than \mathbf{f} for which there is an $\mathbf{x} <_{\mathcal{I}} \mathbf{f}$ so the only light minimal \mathcal{I} -degrees less than \mathbf{x} are \mathbf{c} and \mathbf{d} , and \mathbf{x} has a light \mathcal{I} -strongly minimal cover \mathbf{y} , which in turn has a light \mathcal{I} -strongly minimal cover \mathbf{z} which is $\leq_{\mathcal{I}} \mathbf{f}$.

Lemma 7.2. *Let $F = \{\{\mathbf{a}_i, \mathbf{b}_i\} \mid i < n\}$ be a finite set of pairs of light minimal \mathcal{I} -degrees, so that the \mathbf{a}_i 's and the \mathbf{b}_j 's are distinct. Then there is a name for F .*

Proof. In this proof we use that in $\mathbf{Ceers}_{/\mathcal{I}}$ every non-universal element has infinitely many distinct self-full strong minimal covers, see [4, Theorem 7.9]. For each pair $\{\mathbf{a}_i, \mathbf{b}_i\} \in F$, we let \mathbf{c}_i be a self-full \mathcal{I} -strongly minimal cover of $\mathbf{a}_i \oplus \mathbf{b}_i$. Let \mathbf{d}_i be a self-full \mathcal{I} -strongly minimal cover of \mathbf{c}_i . Note that we choose these to be \mathcal{I} -strongly minimal covers in $\mathbf{Ceers}_{/\mathcal{I}}$, not just in $\mathbf{Light}_{/\mathcal{I}}$. Let $\mathbf{f} = \bigoplus_i \mathbf{d}_i$.

First we check that for each pair $\{\mathbf{a}_i, \mathbf{b}_i\}$ in F , there is an $\mathbf{x}_i <_{\mathcal{I}} \mathbf{f}$ (take $\mathbf{x}_i = \mathbf{a}_i \oplus \mathbf{b}_i$) so that the only light minimal \mathcal{I} -degrees less than \mathbf{x}_i are \mathbf{a}_i and \mathbf{b}_i , and \mathbf{x}_i has a light \mathcal{I} -strongly minimal cover \mathbf{y}_i (take $\mathbf{y}_i = \mathbf{c}_i$) which has a light \mathcal{I} -strongly minimal cover $\mathbf{z}_i \leq_{\mathcal{I}} \mathbf{f}$ (take $\mathbf{z}_i = \mathbf{d}_i$). To see that the only light minimal \mathcal{I} -degrees $\leq_{\mathcal{I}} \mathbf{x}_i$ are \mathbf{a}_i and \mathbf{b}_i , assume that $A_i \in \mathbf{a}_i$, $B_i \in \mathbf{b}_i$, and $X_i = A_i \oplus B_i$. If $U \leq_{\mathcal{I}} X_i$ has light minimal \mathcal{I} -degree then by Fact 2.7 there exist U_0, U_1 such that $U \equiv_{\mathcal{I}} U_0 \oplus U_1$ with $U_0 \leq A_i$ and $U_1 \leq B_i$. By light minimality of A_i, B_i it follows that U_0 is either finite, dark, $U_0 \equiv_{\mathcal{I}} A_i$, or $U_0 \equiv_{\mathcal{I}} \text{Id}$, and similarly U_1 is either finite, dark, $U_1 \equiv_{\mathcal{I}} B_i$, or $U_1 \equiv_{\mathcal{I}} \text{Id}$. If $U_0 \equiv_{\mathcal{I}} A_i$, then we see that $A_i \leq_{\mathcal{I}} U$, so by light minimality of U and A_i , we have that $U \equiv_{\mathcal{I}} A_i$. Similarly if $U_1 \equiv_{\mathcal{I}} B_i$ then $U \equiv_{\mathcal{I}} B_i$. Thus we can assume neither of these cases holds. By lightness of U , it follows at least one of U_0 or U_1 is light, so without loss of generality, we suppose U_0 is light, i.e. $U_0 \equiv_{\mathcal{I}} \text{Id}$. If U_1 is finite or $\equiv_{\mathcal{I}} \text{Id}$, then $U \equiv_{\mathcal{I}} \text{Id}$, contradicting U being of light minimal \mathcal{I} -degree. Thus U_1 must be dark $\leq B_i$. So, $U \equiv_{\mathcal{I}} \text{Id} \oplus D$ for some dark $D \leq B_i$. But since every dark ceer D has a join with Id , namely $D \oplus \text{Id}$ [4, Obs 5.1], it follows that $U \equiv_{\mathcal{I}} D \oplus \text{Id} \leq B_i$. Again, by light minimality of U and B_i , we see $U \equiv_{\mathcal{I}} B_i$.

It remains to show that no other pair $\leq_{\mathcal{I}} \mathbf{f}$ has such a triple $\mathbf{x}, \mathbf{y}, \mathbf{z}$. We begin with an easy observation about the \mathcal{I} -degrees $\leq \mathbf{d}_i$. Together with $A_i \in \mathbf{a}_i$, $B_i \in \mathbf{b}_i$, fix also representatives $C_i \in \mathbf{c}_i$, $D_i \in \mathbf{d}_i$, and $f \in \mathbf{f}$. We also use the notation $\mathbf{0}_{\mathcal{I}}$ to denote the least \mathcal{I} -degree, which is the $\equiv_{\mathcal{I}}$ -class of the ceer Id_1 with only one equivalence class, and is comprised exactly of all finite ceers.

Next we show that for each \mathbf{a}_i there can be at most one dark \mathcal{I} degree $\mathbf{r}_i \leq_{\mathcal{I}} \mathbf{a}_i$. To see this assume towards a contradiction that \mathbf{r}, \mathbf{s} are distinct dark \mathcal{I} -degrees below \mathbf{a}_i . By [4, Obs 5.1 and Lemma 6.5], $\mathbf{r} \oplus \text{Id}$ and $\mathbf{s} \oplus \text{Id}$ are joins of the two \mathcal{I} -degrees, and thus they are both $\leq_{\mathcal{I}} \mathbf{a}_i$. Since ι is an embedding, $\mathbf{r} \oplus \text{Id}$ and $\mathbf{s} \oplus \text{Id}$ are distinct light degrees in $(\text{Id}, \mathbf{a}_i]$. This contradicts the light minimality of \mathbf{a}_i . We conclude that there can be only one dark \mathcal{I} -degree $\leq_{\mathcal{I}} \mathbf{a}_i$. Further, if there is a (unique) dark \mathcal{I} -degree $\mathbf{r}_i \leq_{\mathcal{I}} \mathbf{a}_i$ then $\mathbf{a}_i = \mathbf{r}_i \oplus \text{Id}$. In the following if there is no dark degree below \mathbf{a}_i then we let $\mathbf{r}_i = \mathbf{a}_i$. Similar considerations and notations hold for each \mathbf{b}_i . In particular, if there is a (unique) dark degree \mathcal{I} -degree $\leq \mathbf{b}_i$, we call it \mathbf{s}_i , and if there is no dark $\mathbf{s}_i \leq_{\mathcal{I}} \mathbf{b}_i$, then we let $\mathbf{s}_i = \mathbf{b}_i$.

Claim 7.3. *If $\mathbf{w} \leq_{\mathcal{I}} \mathbf{d}_i$ is an \mathcal{I} -degree, then $\mathbf{w} \in \{\mathbf{0}_{\mathcal{I}}, \text{Id}, \mathbf{a}_i, \mathbf{b}_i, \mathbf{r}_i, \mathbf{s}_i, \mathbf{r}_i \oplus \mathbf{b}_i, \mathbf{a}_i \oplus \mathbf{s}_i, \mathbf{a}_i \oplus \text{Id}, \mathbf{b}_i \oplus \text{Id}, \mathbf{a}_i \oplus \mathbf{b}_i, \mathbf{c}_i, \mathbf{d}_i\}$.*

Proof. If $W \leq_{\mathcal{I}} D_i$, then $W \equiv_{\mathcal{I}} D_i$ or $W \equiv_{\mathcal{I}} C_i$ or $W \leq_{\mathcal{I}} A_i \oplus B_i$ since \mathbf{d}_i is an \mathcal{I} -strongly minimal cover of \mathbf{c}_i which is an \mathcal{I} -strongly minimal cover of $\mathbf{a}_i \oplus \mathbf{b}_i$. If $W \leq_{\mathcal{I}} A_i \oplus B_i$, then by Fact 2.7 $W \equiv_{\mathcal{I}} W_0 \oplus W_1$ where $W_0 \leq A_i$ and $W_1 \leq B_i$. Since each \mathbf{a}_i and \mathbf{b}_i is light minimal, the only possibilities for the \mathcal{I} -degree of W_0 are $\mathbf{a}_i, \mathbf{r}_i, \text{Id}$, or $\mathbf{0}_{\mathcal{I}}$, and the only possibilities for the \mathcal{I} -degree of W_1 are $\mathbf{b}_i, \mathbf{s}_i, \text{Id}$, or $\mathbf{0}_{\mathcal{I}}$. The possible direct sums of these are easily seen to be the possibilities listed in the claim. \square

Suppose that the only light minimal \mathcal{I} -degrees bounded by \mathbf{x} are \mathbf{a}_i and \mathbf{a}_j , with $i \neq j$ (it is no different if we consider \mathbf{a}_i and \mathbf{b}_j or \mathbf{b}_i and \mathbf{b}_j). Since $X \leq_{\mathcal{I}} \mathbf{f} \equiv_{\mathcal{I}} \bigoplus D_r$, we see by Fact 2.7 that $X \equiv_{\mathcal{I}} \bigoplus_r X_r$ where each $X_r \leq D_r$. Thus each X_r is \mathcal{I} -equivalent to one of the \mathcal{I} -degrees in the list: $\mathbf{0}_{\mathcal{I}}, \text{Id}, \mathbf{a}_r, \mathbf{b}_r, \mathbf{r}_r, \mathbf{s}_r, \mathbf{r}_r \oplus \mathbf{b}_r, \mathbf{a}_r \oplus \mathbf{s}_r, \mathbf{a}_i \oplus \text{Id}, \mathbf{b}_i \oplus \text{Id}, \mathbf{a}_r \oplus \mathbf{b}_r, \mathbf{c}_r, \mathbf{d}_r$. Since \mathbf{x} only bounds \mathbf{a}_i and \mathbf{a}_j , for the \mathcal{I} -degrees of the various X_r we must rule out the possibilities that any such degree bounds in the above list a light minimal \mathcal{I} -degree not in $\{\mathbf{a}_i, \mathbf{a}_j\}$; finally we can remove from the list the \mathcal{I} -degrees $\mathbf{a}_i \oplus \mathbf{s}_i$, and $\mathbf{a}_j \oplus \mathbf{s}_j$: note for example that $\mathbf{a}_i \oplus \mathbf{s}_i \geq_{\mathcal{I}} \text{Id} \oplus \mathbf{s}_i = \mathbf{b}_i$, so this one cannot be the \mathcal{I} -degree of X_i , unless $\mathbf{b}_i \in \{\mathbf{a}_i, \mathbf{a}_j\}$. Then we conclude that the \mathcal{I} -degree of any of the X_r 's

is equal to one of the \mathcal{I} -degrees in the following list: $\mathbf{0}_{\mathcal{I}}, \mathbf{Id}, \mathbf{a}_i, \mathbf{r}_k, \mathbf{s}_k, \mathbf{a}_i \oplus \mathbf{Id}, \mathbf{a}_j, \mathbf{a}_j \oplus \mathbf{Id}$, for some k .

We write \mathbf{a}_i as either \mathbf{r}_i or $\mathbf{r}_i \oplus \mathbf{Id}$, depending on which is true. Similarly for \mathbf{a}_j . It then follows that the only possibility for \mathbf{x} is to be of the form $\mathbf{r}_i \oplus \mathbf{r}_j (\oplus \mathbf{r}_k) (\oplus \mathbf{s}_k) (\oplus \mathbf{Id})$ where parentheses are meant to symbolize that we may or may not be joining with these degrees. But we observe that any degree $\mathbf{u} \geq_{\mathcal{I}} \mathbf{a}_i$ and $\geq_{\mathcal{I}} \mathbf{r}_k$ for any k , is also $\geq_{\mathcal{I}} \mathbf{a}_k$. This is because \mathbf{u} is light, and thus $\geq_{\mathcal{I}} \mathbf{a}_k$, since \mathbf{a}_k is the join of \mathbf{Id} and \mathbf{r}_k . Since \mathbf{x} bounds only \mathbf{a}_i and \mathbf{a}_j , this is impossible if $k \notin \{i, j\}$. Thus, we are left with the only possibilities for \mathbf{x} being $\mathbf{r}_i \oplus \mathbf{r}_j (\oplus \mathbf{Id})$. Now, \mathbf{x} bounded \mathbf{a}_i and \mathbf{a}_j and no other light minimal \mathcal{I} -degrees, and \mathbf{y} was an \mathcal{I} -strongly minimal cover of \mathbf{x} , then it would also bound no other light minimal degrees, and \mathbf{z} being an \mathcal{I} -strongly minimal cover of \mathbf{y} would also bound no other light minimal \mathcal{I} -degrees. On the other hand, the same argument as before applied to \mathbf{y} and \mathbf{z} would show that also \mathbf{y} and \mathbf{z} may only have the form $\mathbf{r}_i \oplus \mathbf{r}_j (\oplus \mathbf{Id})$. But there are at most two such \mathcal{I} -degrees, contradicting the fact that $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are three distinct \mathcal{I} -degrees.

Thus \mathbf{f} is a name for F . □

Definition 7.4. Given two good codes \mathbf{c}, \mathbf{c}' for graphs in $\mathbf{Light}_{/\mathcal{I}}$ (thus giving models of Robinson's Q), we say that a pair of names (\mathbf{f}, \mathbf{g}) is a *label* for a partial function $H : H_{\mathbf{c}} \rightarrow H_{\mathbf{c}'}$ if \mathbf{f} is a name for a set F and \mathbf{g} is a name for a set G and whenever $\mathbf{a} <_{\mathcal{I}} \mathbf{c}$ is light minimal, then there is a light minimal \mathcal{I} -degree \mathbf{b} so that $\{\mathbf{a}, \mathbf{b}\}$ is a pair in F and $\mathbf{b} \not\leq_{\mathcal{I}} \mathbf{c}, \mathbf{c}'$, and $\{\mathbf{b}, H(\mathbf{a})\}$ is a pair in G .

Lemma 7.5. *If H is a finite function between $H_{\mathbf{c}}$ and $H_{\mathbf{c}'}$, with \mathbf{c}, \mathbf{c}' light \mathcal{I} -degrees, there exists a pair (\mathbf{f}, \mathbf{g}) which is a label for this function.*

Proof. We need to use light minimal degrees which are not below $\mathbf{c} \oplus \mathbf{c}'$ to interpolate for the function. Such degrees exist, because there are infinitely many dark minimal degrees avoiding any lower cone [4, Theorem 3.3]. By Lemma 6.2 we just use the ι image of these. If C, C' are representatives of \mathbf{c}, \mathbf{c}' respectively, and $D \not\leq C \oplus C'$ is dark minimal, then $\iota(D) \not\leq_{\mathcal{I}} C \oplus C'$ because $D \not\leq_{\mathcal{I}} C \oplus C'$, by Observation 3.4

The existence of the names for the needed sets of pairs is then given by Lemma 7.2. □

Definition 7.6. On \mathcal{I} -degrees we define the equivalence relation $(\mathbf{c}, \mathbf{d}) \sim (\mathbf{c}', \mathbf{d}')$ if the two pairs coincide, or \mathbf{c} and \mathbf{c}' are good codes, $\mathbf{d} \in U^{\mathbf{c}}$ and $\mathbf{d}' \in U^{\mathbf{c}'}$ and there exists a pair of names (\mathbf{f}, \mathbf{g}) which is a label for a function H which is an order-preserving bijection between $[\mathbf{0}^{\mathbf{c}}, \mathbf{d}]^{U^{\mathbf{c}}}$ and $[\mathbf{0}^{\mathbf{c}'}, \mathbf{d}']^{U^{\mathbf{c}'}}$, where the various symbols $U^{\mathbf{c}}, U^{\mathbf{c}'}, \mathbf{0}^{\mathbf{c}}$ and $\mathbf{0}^{\mathbf{c}'}$ have the same meanings as in Section 5.

This is what is needed to again define \mathbb{N} exactly as in the dark case, as explained in Section 5.

Thus we have shown that there is a copy of $(\mathbb{N}, +, \times)$ interpretable without parameters in the structure $\mathbf{Light}_{/\mathcal{I}}$, proving the following theorem:

Theorem 7.7. *The first order theory of $\mathbf{Light}_{/\mathcal{I}}$ is computably isomorphic to true first order arithmetic.*

Corollary 7.8. *The first order theory of \mathbf{Light} is computably isomorphic to true first order arithmetic.*

Proof. \mathcal{I} -equivalence on light \mathcal{I} -degrees is \emptyset -definable in the light degrees, by Lemma 2.8. □

8. OPEN QUESTIONS

The argument in [2] showed that the 3-quantifier theory of **Ceers** is already undecidable, but does not clarify its degree. We ask:

Question 8.1. *What is the degree of the 3-quantifier theory of **Ceers**?*

Question 8.2. *What is the least n so that the n -quantifier theory of **Ceers** is undecidable?*

Our interpretation of $(\mathbb{N}, +, \cdot)$ shows that there is some k (for example any k so that the interpretation is \exists_k in the hierarchy of formulas, as described for instance in [18, p.47f]) so that for every m , the $k + m$ -quantifier theory of **Ceers** is $\geq_T \emptyset^{(m)}$. We ask:

Question 8.3. *What is the least such k ? Is it true that the $k + m$ -quantifier theory of **Ceers** is $\equiv_T \emptyset^{(m)}$?*

In analyzing the \mathbb{Z} -dark minimal ceers, we concluded that the \mathcal{I} -degrees of a \mathbb{Z} -dark minimal ceer all of whose classes are computable have joins with every other \mathcal{I} -degree.

In [4], we showed that Id is \emptyset -definable in **Ceers** as the only ceer which is minimal over the finite ceers and has a join with every other ceer. Lemma 3.12 shows that in **Ceers** $_{/\mathcal{I}}$, any \mathbb{Z} -dark minimal ceer with all computable classes also has this property. We ask:

Question 8.4. *Is Id \emptyset -definable in **Ceers** $_{/\mathcal{I}}$?*

Is the collection of \mathcal{I} -degrees of dark minimal ceers definable (even with parameters)? That is, is there a definable way to distinguish between dark minimal and \mathbb{Z} -dark minimal \mathcal{I} -degrees?

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