

RECURSIVE SPECTRA OF FLAT STRONGLY MINIMAL THEORIES

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1. INTRODUCTION

ABSTRACT. We show that for a model complete strongly minimal theory whose pregeometry is flat, the recursive spectrum ($\text{SRM}(T)$) is either of the form $[0, \alpha)$ for $\alpha \in \omega + 2$ or $[0, n) \cup \{\omega\}$ for $n \in \omega$, or contained in $\{0, 1, 2, \omega\}$.

Combined with previous results, this leaves precisely 8 sets for which it is not yet determined whether they are spectra of a model complete strongly minimal theory with a flat pregeometry.

Recursive model theory examines what information is required to recursively present models of a given theory. Of particular interest is the case that the theory itself poses no obstacles to computing its models. This is exemplified by the case where a theory T has both recursive and non-recursive models. The spectrum problem for strongly minimal theories focuses on the clearest case of this phenomenon. From this point onward, we only consider complete theories. The Baldwin-Lachlan theorem [BL71] shows that the countable models of a strongly minimal non- \aleph_1 -categorical theory T are characterized by dimension and thus T has a countable chain of countable models: $\mathcal{M}_0 \preceq \mathcal{M}_1 \preceq \cdots \preceq \mathcal{M}_\omega$. We let the spectrum of recursive models of T , or $\text{SRM}(T)$, be the set of $k \in \omega + 1$ so that \mathcal{M}_k has a recursive presentation. The spectrum problem asks to characterize the sets $S \subseteq \omega + 1$ which occur as $\text{SRM}(T)$. The spectrum problem has been a focus in recursive model theory since Goncharov [Gon78] first showed the existence of a non-trivial spectrum.

It has proven useful [GHL⁺03, And14, AM14] to examine the spectrum problem in light of the pregeometry of the theory. Zilber famously conjectured a trichotomy for the possible pregeometries of strongly minimal theories: That the pregeometry is either disintegrated, locally modular, or field-like. Hrushovski [Hru93] produced a counterexample, thus introducing a new type of pregeometry: The flat pregeometries. In particular, flat pregeometries are non-disintegrated and preclude the theory from interpreting a group, so flat pregeometries do not fit into Zilber's conjectured trichotomy. Partial results have been obtained for the spectrum problem in the case of each of the trichotomous pregeometries. For example, Andrews and Medvedev [AM14] show that the only spectra of disintegrated strongly minimal theories with finite signatures are $\emptyset, \{0\}, \omega + 1$. They also show that if T is a locally modular strongly minimal theory with a finite signature which expands a group, then $\text{SRM}(T)$ is among $\emptyset, \{0\}, \omega + 1$. They also note that field-like strongly minimal theories with finite signatures expanding a field have SRM of $\omega + 1$.

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So far, we have very little information about the spectra of flat strongly minimal theories, i.e., strongly minimal theories with flat pregeometries. Each of these other cases: Disintegrated theories, locally modular theories expanding a group, or field-like theories expanding a field admit some level of quantifier-elimination after naming parameters. This is not true for flat strongly minimal theories (e.g., [And13]). In this paper, we examine the spectrum problem for *model complete* flat strongly minimal theories. This is a wide enough class to encompass all the existing constructions of spectra of a flat strongly minimal theory.

In particular, we show:

Theorem 1. *If T is a model complete flat strongly minimal theory, then $\text{SRM}(T)$ is of one of the following forms:*

- $[0, \alpha)$ for $\alpha \in \omega + 2$, $[0, n] \cup \{\omega\}$ for $n \in \omega$, $\{\omega\}$,
- $\{1\}$, $\{2\}$, $\{0, 2\}$, $\{1, 2\}$, $\{1, \omega\}$, $\{2, \omega\}$, $\{0, 2, \omega\}$, $\{1, 2, \omega\}$.

Further, the sets in the first line are in fact spectra of model complete flat strongly minimal theories. We do not know whether any of the 8 sets (not schema) in the second line are spectra of model complete flat strongly minimal theories.

We note that the proof of this result requires both some recursion theoretic arguments (as in section 3) and some purely geometric arguments (as in sections 4 and 6).

2. BACKGROUND

A *combinatorial pregeometry* G is a set X equipped with an operator $\text{cl} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ satisfying, for every $A \subseteq B \subseteq X$, $a, b \in X$:

- $A \subseteq \text{cl}(A) \subseteq \text{cl}(B) = \text{cl}(\text{cl}(B))$ (Closure operator)
- if $a \in \text{cl}(Ab) \setminus \text{cl}(A)$, then $b \in \text{cl}(Aa)$ (Exchange Principle)
- $\text{cl}(A) = \bigcup \{\text{cl}(A_0) : A_0 \subseteq A, |A_0| < \infty\}$ (Finite character)

We say that A is closed if $A = \text{cl}(A)$. We say that “ a depends on A ” when $a \in \text{cl}(A)$. We say that a set of points A is *independent over* B if $a \notin \text{cl}(B \cup (A \setminus \{a\}))$ for every $a \in A$. We say A is *independent* if it is independent over the empty set. A *basis* for a set A is a minimal $Y \subseteq A$ such that $A \subseteq \text{cl}(Y)$. The *dimension* of a set A , denoted $\text{d}(A)$, is the cardinality of any basis of A — it follows from the exchange principle that all bases have the same cardinality.

A pregeometry is *disintegrated* if whenever b depends on A , then b depends on some single element $a_0 \in A$, i.e., $\text{cl}(A) = \bigcup_{a_0 \in A} \text{cl}(a_0)$.

For \mathcal{N} a strongly minimal structure, we define $G_{\mathcal{N}}$ to be the pregeometry given by the algebraic closure operator $\text{acl}_{\mathcal{N}}$. When the structure \mathcal{N} is clear from context, we drop the subscript. The pregeometry of a strongly minimal theory T is $G_{\mathcal{M}}$, where \mathcal{M} is the countable saturated model of T . In this paper, we will be interested in theories whose pregeometry is *flat*:

Definition 2. In the context of an ambient pregeometry G , let $\Sigma = \{E_1, \dots, E_k\}$ be a collection of finite dimensional closed subsets of G . We denote

$$\Delta(\Sigma) = \sum_{\emptyset \neq s \subseteq \{1, \dots, k\}} (-1)^{|s|+1} \text{d} \left(\bigcap_{i \in s} E_i \right).$$

This is the result of applying the inclusion-exclusion principle to the elements of Σ , where cardinality is replaced with dimension.

A pregeometry G is *flat* if it is non-disintegrated and whenever Σ is a finite collection of finite dimensional closed sets in G , then $\Delta(\Sigma) \geq d(\bigcup \Sigma)$.

Note that we include in the definition of flatness that the pregeometry is non-disintegrated. This choice is not universally made in the literature, but it allows us to concisely focus on the collection of pregeometries we wish to consider.

Definition 3. A pregeometry G is called *homogeneous* if for any set A , for any elements $a, b \in G \setminus \text{cl}(A)$, there is an automorphism σ of G over A with $\sigma(a) = b$. In particular, for any two independent tuples \bar{a}, \bar{b} of the same length, there is an automorphism σ of G with $\sigma(\bar{a}) = \bar{b}$.

The pregeometry of a model of a strongly minimal theory T is homogeneous, because every model of T is homogeneous and all independent tuples of length k have the same type — the unique generic k -type.

The following useful fact follows directly from the Tarski-Vaught test for being an elementary substructure.

Fact 4. *If \mathcal{N} is strongly minimal and $X \subseteq N$ is algebraically closed and infinite, then $X \preceq \mathcal{N}$. It follows that for cardinals $\alpha < \beta$, if a strongly minimal theory T has a model of dimension α , then T has a model of dimension β .*

In examining the spectrum problem, there is a slight disconnect between the indices used in the definition of the spectrum of T and the dimensions of the models of T . It follows from Fact 4 that if p is the dimension of the prime model of T , then \mathcal{M}_k (the k th model in the countable chain of models of T) has dimension $p + k$.

We use the following notation throughout: If \mathcal{N} is a structure, $\bar{a} \in \mathcal{N}$, and $\varphi(\bar{x}, \bar{y})$ is a formula, then $\varphi(\mathcal{N}, \bar{a}) = \{\bar{b} : \mathcal{N} \models \varphi(\bar{b}, \bar{a})\}$.

All signatures are assumed to be recursive. Furthermore, since it makes no difference for recursive presentations, we assume that all signatures are relational.

3. OUR MAIN STRATEGY

The following Lemma is our main tool in showing that if the $k + 1$ -dimensional model is recursive, then the k -dimensional model is recursive.

Lemma 5. *Suppose that \mathcal{N} is a recursive strongly minimal structure and $\mathcal{M} \preceq \mathcal{N}$ is a Δ_2^0 subset of \mathcal{N} . Further suppose that $A \subseteq M$ is any infinite Σ_1^0 set. Then \mathcal{M} has a recursive copy.*

Proof. Let M_s be the elements of N which are in M according to stage s of the Δ_2^0 -approximation to M . We can alter the approximation to ensure that if $x \in A_s$, the elements enumerated into A by stage s , then $x \in M_s$. We build \mathcal{B} in stages, so that at stage s we build \mathcal{B}_s a finite structure in a finite sub-signature \mathcal{L}_s of the signature \mathcal{L} of \mathcal{N} . We copy elements from \mathcal{N} , attempting to only copy the elements in M . At each stage s , we have an embedding f_s from our \mathcal{B}_s into \mathcal{N} . We start with $\mathcal{B}_0 = \emptyset$, $\mathcal{L}_s = \emptyset$, and $f_0 = \emptyset$.

We now describe the stage $s + 1$ -construction: If the range of f_s is contained in M_{s+1} , then we take y to be the least member of M_{s+1} which is not in the range of f_s , and we append a new element x to \mathcal{B}_{s+1} to copy this element. We extend f_s to set $f_{s+1}(x) = y$. If $\mathcal{L}_s \neq \mathcal{L}$, we add another symbol from \mathcal{L} to \mathcal{L}_s to form \mathcal{L}_{s+1} . We then determine atomic facts in \mathcal{B}_{s+1} to maintain that f_{s+1} is an embedding of \mathcal{B}_{s+1} into \mathcal{N} .

If the range of f_s is not contained in M_{s+1} , then we let z be the least element of the range of f_s which is not in M_{s+1} . Let \bar{c} be the elements of $M_{s+1} \cap \text{ran}(f_s)$ and $z\bar{y}$ be the elements of $\text{ran}(f_s) \setminus M_{s+1}$. We wait for a stage $t > s$ such that one of two things happens: Either (Outcome 1) $M_t \cap \text{ran}(f_s) \neq M_{s+1} \cap \text{ran}(f_s)$ or (Outcome 2) there is some element $a \in A_t$ and tuple \bar{y}' so that altering f_s to send $f_s^{-1}(z)$ to a instead of z and $f_s^{-1}(\bar{y})$ to \bar{y}' will still give an embedding of \mathcal{B}_s into \mathcal{N} . We speed-up the construction by doing nothing at each stage between s and t (so $\mathcal{B}_{t-1} = \mathcal{B}_s$, $f_{t-1} = f_s$, and $\mathcal{L}_{t-1} = \mathcal{L}_s$). In the first case, we simply proceed to the next stage with $f_t = f_s$. Note that the first outcome can happen only finitely many consecutive times, since the Δ_2^0 -approximation must settle down on $\text{ran}(f_s)$. In the second case, we let f_t be as described and go to the next stage. In both cases, we keep $\mathcal{B}_t = \mathcal{B}_s$ and $\mathcal{L}_t = \mathcal{L}_s$.

We first argue that one of these two outcomes must happen. Suppose otherwise. In particular, $M_{s+1} \cap \text{ran}(f_s) = M_t \cap \text{ran}(f_s)$ for all $t > s$, so $\bar{c} \subseteq M$. Then consider the formula $\exists \bar{y} \varphi(\bar{c}, z, \bar{y})$ where φ is the formula describing all of our atomic and negation of atomic commitments regarding the finite structure \mathcal{B}_s . By strong minimality, this describes either a finite or co-finite subset of \mathcal{N} . First suppose that it is finite, then $z \in \text{acl}(\bar{c})$. But since $\bar{c} \subseteq M$, this shows that $z \in M$. But $z \notin M_{s+1}$. Thus the approximation to M on $\text{ran}(f_s)$ must change at some point yielding Outcome 1 and a contradiction. Next suppose that it is co-finite. Then there is some $a \in A$ satisfying this formula. This exactly gives us the $a\bar{y}'$ needed to attain Outcome 2, again yielding a contradiction.

We now show that for every $x \in B$, $\lim_s f_s(x)$ converges to a member of M . This is shown by induction. Suppose that all previous elements of B have already had f_s converge to elements of M . Then $f_t(x)$ can only change after stage s by moving to become an element of A_t . This assignment is then permanent since $x \in M_r$ for all later r , due to x being in A_r . Further, if $f_s(x)$ is not a member of M , then at some stage t where the approximation to M has settled down on x , perhaps after finitely many Outcome 1s, we will reassign f on x to be a member of A . Thus we can define $F(x) = \lim_s f_s(x)$.

Finally, we argue that every member $x \in M$ is in the range of F . This is also proved by induction. Let s be a stage so that all previous elements of M are in the range of f_s and that $\lim f_s$ has already converged on these elements. Further, let s be late enough that the approximation to M has settled on x . Since we only change f_t on elements whose images do not appear to be in M at stage t , if $x \in \text{ran}(f_t)$, for any $t > s$, then $x \in \text{ran}(F)$. Since F is well defined on \mathcal{B}_s and $\text{ran}(F) \subseteq M$, we can take some stage $t \geq s$ where $\text{ran}(f_t) \subseteq M_{t+1}$. Then $x \in \text{ran}(f_{t+1})$, so $x \in \text{ran}(F)$. \square

Next we see that in any model complete strongly minimal theory, the k -dimensional model is Δ_2^0 in the $k+1$ -dimensional model. With the previous lemma, this focuses the problem on finding Σ_1^0 subsets of models.

Lemma 6. *Let T be a strongly minimal model complete theory. Let \mathcal{N} be a finite-dimensional model and \bar{b} a generic tuple (possibly empty) in \mathcal{N} . Then $\text{acl}(\bar{b})$ is a Δ_2^0 subset of \mathcal{N} .*

Proof. Note that for any tuple \bar{a} , $d \in \text{acl}(\bar{a})$ if and only if there exists an existential formula φ and natural number n so that $\mathcal{N} \models \varphi(\bar{a}, d) \wedge \neg \exists^n x \varphi(\bar{a}, x)$. By model completeness of T , our restriction to existential formulas gives the full algebraic

closure. This shows that $\{d, \bar{a} \mid d \in \text{acl}(\bar{a})\}$ is a Σ_2^0 subset of $\mathcal{N}^{<\omega}$. Thus for any tuple \bar{a} , $\text{acl}(\bar{a})$ is Σ_2^0 .

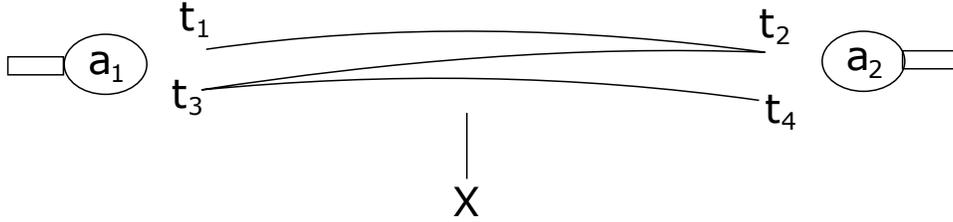
We extend \bar{b} to a basis $\bar{b}, c_1, c_2, \dots, c_k$ of \mathcal{N} . By repeatedly using the exchange property, $x \in \text{acl}(\bar{b})$ if and only if $c_k \notin \text{acl}(\bar{b}, x, c_1, \dots, c_{k-1}) \wedge \dots \wedge c_1 \notin \text{acl}(\bar{b}, x)$. As this is a conjunction of Π_2^0 -conditions, $\text{acl}(\bar{b})$ is also Π_2^0 . \square

Due to the previous two Lemmas, we now turn towards finding infinite Σ_1^0 subsets of models of model complete flat strongly minimal theories. This will rely heavily on the geometric assumption of flatness. Towards this end, we introduce the purely geometric construction of ping-pong sequences. We then show that ping-pong sequences can be found inside the algebraic closure of a generic tuple via a single formula. The core use of geometry is in showing that the ping-pong sequences must be injective, thus giving an infinite set. This will give us infinite Σ_1^0 sets.

We note that this strategy hinges on a strong form of non-local-finiteness. In particular, we will have a single formula φ so that, starting from a finite tuple, repeatedly using φ to generate algebraicities will give us an infinite set. Note that there are disintegrated strongly minimal theories where this cannot be done even with algebraic closures being infinite, such as the constructions employed in [KNS97, Nie99, HKS06]. Similarly, there are locally modular strongly minimal theories where this cannot be done, such as the theory of a vector space over \mathbb{F}_5^{alg} . Thus, this strategy requires the geometric assumption of flatness.

4. INFINITE FORMULA-CLOSURES

4.1. Ping pong sequences. In the context of a non-disintegrated pregeometry G , we describe a procedure for generating a sequence $(t_i : i < n)$, for some $n \in \mathbb{N} \cup \{\omega\}$. Choose arbitrarily a set $X \subseteq G$ and points $a_1, a_2 \in G$ independent over X . For every $Y \subseteq G$ denote $\text{cl}_X(Y) = \text{cl}(X \cup Y)$. Choose arbitrarily $t_1 \notin \text{cl}_X(a_1 a_2)$. For each $i \in \omega$, let $j_i \in \{1, 2\}$ be such that $i \equiv j_i \pmod{2}$. If possible, choose $t_{i+1} \in \text{cl}_X(a_{j_i} t_i) \setminus \text{cl}_X(a_{j_i})$ distinct from t_i . Call such a sequence a *ping-pong sequence*, or PPS for short.



Observation 7. For each $i \in \omega$, if t_{i+1} exists, then $t_{i+1} \notin \text{cl}_X(a_1 a_2)$.

Proof. For $i = 1$, t_1 was chosen so that $t_1 \notin \text{cl}_X(a_1 a_2)$. Assume for a moment there is some i such that $t_i \notin \text{cl}_X(a_1 a_2)$, but $t_{i+1} \in \text{cl}_X(a_1 a_2)$. By construction, using the exchange property, $t_i \in \text{cl}_X(a_{j_i} t_{i+1}) \subseteq \text{cl}_X(a_1 a_2)$, in contradiction. So $t_i \notin \text{cl}_X(a_1 a_2)$ for every $i \in \omega$. \square

Lemma 8. In a flat pregeometry G , a PPS is injective. That is, if t_{i_1}, t_{i_2} are elements of a PPS with $i_1 \neq i_2$, then $t_{i_1} \neq t_{i_2}$.

Proof. A consecutive subsequence of a PPS is also a PPS, so it is enough to prove the statement for $i_1 = 1$. Assume to the contrary that the statement is false. Let l be minimal such that there exists a PPS t_1, \dots, t_{l+1} with $t_1 = t_{l+1}$. Fix a PPS t_1, \dots, t_{l+1} generated by X , a_1, a_2 in a flat pregeometry G witnessing this.

Fix $X_0 \subseteq X$ a basis for X . For each $i < l + 1$, let $F_i = X_0 \cup \{a_{j_i}\}$ and let $E_i = \text{cl}_X(a_{j_i}t_i) = \text{cl}(F_i t_i)$. Let $\Sigma = \{E_i : i < l + 1\}$. We will find an upper bound on $d(\bigcup \Sigma)$ using flatness of G . For this, we need to understand the dimensions of intersections of the form $\bigcap_{i \in s} E_i$, for s a set of indices.

Claim. For a set of indices $s \subseteq \{1, \dots, l\}$ with $|s| \geq 2$ and not of the form $\{i, i + 1\}$ or $\{1, l\}$,

$$\bigcap_{i \in s} E_i = \text{cl} \left(\bigcap_{i \in s} F_i \right)$$

Proof. Fix some $1 \leq m < k < l + 1$. We first show that $E_m \cap E_k \subseteq \text{cl}_X(a_1 a_2)$ unless $\{k, m\}$ equals $\{i, i + 1\}$ or $\{1, l\}$. Towards the contrapositive, let $E_m \cap E_k \not\subseteq \text{cl}_X(a_1 a_2)$. We will show that either $k = m + 1$, or $m = 1$ and $k = l$. Let $t \in E_k \cap E_m \setminus \text{cl}_X(a_1 a_2)$. If $t = t_m$, then t_m, \dots, t_k, t_m is a PPS, contradicting minimality of l unless $m = 1$ and $k = l$, so we may assume $t \neq t_m$. We split into cases, depending on whether m and k have the same parity. Without loss of generality, assume $m \equiv 1 \pmod{2}$.

Case $k \equiv 1 \pmod{2}$: Then $t \in E_k = \text{cl}_X(a_1 t_{k+1})$, implying $t_{k+1} \in \text{cl}_X(a_1 t) \subseteq E_m$. Unless $t_{k+1} = t_m$, which can only happen if $m = 1$ and $k = l$, the sequence $t_1, \dots, t_m, t_{k+1}, \dots, t_{l+1}$ is a PPS contradicting the minimality of l .

Case $k \equiv 0 \pmod{2}$: We may assume $t \neq t_{k+1}$, for otherwise by exchange $E_m = \text{cl}_X(a_1 t) = E_{k+1}$ and $t_{k+1}, t_{m+1}, \dots, t_k, t_{k+1}$ is a PPS, which by minimality of l implies $m = 1$ and $k = l$. Since $t \in E_k = \text{cl}_X(a_2 t_{k+1})$ implies $t_{k+1} \in \text{cl}_X(a_2 t)$, as $t \in E_m$ and $t \neq t_{k+1}$, the sequence $t_1, \dots, t_m, t, t_{k+1}, \dots, t_{l+1}$ is a PPS. By minimality of l , this means that $k = m + 1$.

By what we've shown, for a set $s \subseteq \{1, \dots, l\}$ of size at least 2 that is not of the form $\{i, i + 1\}$ or $\{1, l\}$, there are $m, k \in s$ such that $E_m \cap E_k \subseteq \text{cl}_X(a_1 a_2)$. Thus $\bigcap_{i \in s} E_i = \text{cl}_X(a_1 a_2) \cap \bigcap_{i \in s} E_i = \bigcap_{i \in s} \text{cl}_X(a_1 a_2) \cap E_i$.

For every i , as $\text{cl}_X(a_{j_i}) \subseteq E_i \cap \text{cl}_X(a_1 a_2) \subsetneq E_i$ and $d(E_i) = d(X a_{j_i}) + 1$, it must be that $E_i \cap \text{cl}_X(a_1 a_2) = \text{cl}_X(a_{j_i})$. Thus, taking s as in the claim we have

$$\bigcap_{i \in s} E_i = \bigcap_{i \in s} \text{cl}_X(a_{j_i}) = \text{cl}_X \left(\bigcap_{i \in s} \{a_{j_i}\} \right) = \text{cl} \left(\bigcap_{i \in s} F_i \right).$$

The penultimate inequality is by a_1 and a_2 being independent over X . \square

We now need to address sets of indices s that do not fall under the claim. For $s = \{i\}$, where $i < l + 1$, we already know $d(E_i) = |F_i| + 1$.

Consider $s = \{i, i + 1\}$ for some $i < l$. Because t_{i+1}, a_1, a_2 are independent over X , we know $a_{j_i} \notin E_{i+1}$, so in particular $E_i \cap E_{i+1} \neq E_i$. Now, $X t_{i+1} \subseteq E_i \cap E_{i+1} \subsetneq E_i$ and $d(E_i) = d(X t_{i+1}) + 1$, hence $E_i \cap E_{i+1} = \text{cl}(X t_{i+1})$. Thus, $d(E_i \cap E_{i+1}) = d(X) + 1 = |F_i \cap F_{i+1}| + 1$.

Consider $s = \{1, l\}$. If $l \equiv 2 \pmod{2}$, because $t_{l+1} = t_1$, then $E_l = \text{cl}_X(a_2 t_l) = \text{cl}_X(a_2 t_{l+1}) = \text{cl}_X(a_2 t_1)$ and $E_1 = \text{cl}_X(a_1 t_1)$, so $d(E_l \cap E_1) = |F_l \cap F_1| + 1$ is proved exactly as the case $s = \{i, i + 1\}$. If $l \equiv 1 \pmod{2}$, then $E_1 = E_l = \text{cl}_X(a_1 t_1)$ and $F_1 = F_l$, so again $d(E_1 \cap E_l) = d(E_1) = |F_1| + 1 = |F_1 \cap F_l| + 1$.

Due to the above, we will see some cancellation in the alternating sum we are about to compute as the bound to $d(\bigcup_{i=1}^l E_i)$. For each $i < l$ we have $d(E_i) - d(E_i \cap E_{i+1}) = |F_i| - |F_i \cap F_{i+1}|$, and for $i = l$, we have $d(E_l) - d(E_1 \cap E_l) = |F_l| - |F_1 \cap F_l|$. Therefore, using this observation and the claim, and then the inclusion-exclusion principle,

$$\sum_{\emptyset \neq s \subseteq \{1, \dots, l\}} (-1)^{|s|+1} d\left(\bigcap_{i \in s} E_i\right) = \sum_{\emptyset \neq s \subseteq \{1, \dots, l\}} (-1)^{|s|+1} \left| \bigcap_{i \in s} F_i \right| = \left| \bigcup_{i=1}^l F_i \right|.$$

But $\bigcup_{i=1}^l F_i = X_0 \cup \{a_1, a_2\}$, so by flatness, $d\left(\bigcup_{i=1}^l E_i\right) \leq |X_0 \cup \{a_1, a_2\}| = d(Xa_1a_2)$. Since $t_1 \in \bigcup_{i=1}^l E_i$, this is a contradiction to $t_1 \notin \text{cl}_X(a_1a_2)$, which proves the lemma. \square

4.2. Formula-closures. We define the formula-closure of a set.

Definition 9. Let \mathcal{N} be a model, $X \subseteq \mathcal{N}$ a set, and $\psi(x_1, \dots, x_m)$ some formula. Let $X_\psi^0 = X$ and recursively define

$$X_\psi^{i+1} = \bigcup_{\substack{a_1, \dots, a_{m-1} \in X_\psi^i \\ 0 \leq j < m}} \psi(a_1, \dots, a_j, \mathcal{N}, a_{j+1}, \dots, a_{m-1}).$$

We define the ψ -closure of X in \mathcal{N} to be $\Lambda_\psi^{\mathcal{N}}(X) = \bigcup_{i \in \omega} X_\psi^i$.

We now turn to choosing a formula φ so that $\Lambda_\varphi^{\mathcal{N}}$ is algebraic and, in many cases, will be infinite.

Definition 10. In a pregeometry, a *circuit* is a dependent tuple whose every proper subtuple is independent.

Definition 11. For T a strongly minimal theory whose pregeometry is non-disintegrated, we let $\varphi_0(x_1, \dots, x_m)$ be a formula which witnesses the smallest circuit of size > 2 in the pregeometry of T , i.e., φ_0 holds on some circuit of size m in the countable saturated model of T , but not on a generic m -tuple. The existence of φ_0 follows from non-disintegration.

We let φ be a formula which witnesses the smallest circuit of size > 2 in the pregeometry of T so that φ makes each coordinate algebraic over the others. That is, for each $1 \leq j \leq m$, $T \models \forall \bar{x} \exists^{< \infty} y \varphi(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_m)$. Note that choosing $\varphi(\bar{x})$ to be $\varphi_0(\bar{x}) \wedge \bigwedge_{j \leq m} \exists^{< \infty} y \varphi_0(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_m)$ suffices, as every circuit satisfying φ_0 also satisfies φ . That our choice of φ can be expressed as a first order formula follows from strong minimality of T .

We define n to be $m - 1$. This is the dimension of the smallest circuit of size > 2 , which will be more natural to use below than the size of the circuit.

The next observation and lemma are in terms of T , φ , n as in Definition 11.

Observation 12. By choice of φ , since $\varphi(a_1, \dots, a_{i-1}, \mathcal{N}, a_{i+1}, a_m)$ is always finite in $\mathcal{N} \models T$, for every $X \subseteq \mathcal{N}$ we have $\Lambda_\varphi^{\mathcal{N}}(X) \subseteq \text{acl}_{\mathcal{N}}(X)$.

Lemma 13. Let $\mathcal{N} \models T$ and suppose that the pregeometry of T is flat. If $Y \subseteq \mathcal{N}$ contains $n + 1$ independent points, then $\Lambda_\varphi^{\mathcal{N}}(Y)$ is infinite.

Proof. Let $t_1, a_1, a_2, y_1, \dots, y_{n-2} \in Y$ be independent. Beginning with t_1 and using a_1, a_2 , and $\bar{X} = (y_1, \dots, y_{n-2})$, we define a PPS contained in $\Lambda_\varphi^{\mathcal{N}}(Y)$. Assuming we were able to construct $t_1, \dots, t_i \notin \text{acl}(\bar{X}a_1a_2)$, we only need to show that an appropriate t_{i+1} exists.

Since T is strongly minimal, every independent n -tuple in \mathcal{N} has the same type. In particular, by choice of φ , the independent n -tuple (t_i, a_{j_i}, \bar{X}) can be extended to a circuit which is an instance of φ by an element t_{i+1} . By virtue of $(t_i, a_{j_i}, \bar{X}, t_{i+1})$ being a circuit, $t_{i+1} \in \text{acl}_{\bar{X}}(a_{j_i}t_i) \setminus \text{acl}_{\bar{X}}(a_{j_i})$.

We get a PPS $(t_i : i \in \omega)$, which by Lemma 8 contains infinitely many elements. By construction, clearly $t_i \in \Lambda_\varphi^{\mathcal{N}}(Y)$ for every $i \in \omega$. \square

5. RECURSION THEORETIC CONSEQUENCES

Fix T a model complete flat strongly minimal theory. We let φ and n be as defined in Definition 11. Suppose that \mathcal{N} is a finite-dimensional recursive model of T . In this section, we will show that every model $\mathcal{M} \preceq \mathcal{N}$ of dimension at least n has a recursive presentation. We first show the result for models \mathcal{M} of dimensions at least $n + 1$.

Theorem 14. *Suppose that \mathcal{N} is a finite dimensional recursive model of T . Then every model of T with dimension $k \in [n + 1, \dim(\mathcal{N})]$ has a recursive presentation.*

Proof. Fix a generic k -tuple \bar{x} in \mathcal{N} . By Lemma 6, $\text{acl}(\bar{x})$ is a Δ_2^0 subset of \mathcal{N} . By model completeness of \mathcal{N} , φ defines a recursive subset of N^{n+1} , as both it and its negation are existentially defined by model completeness. It follows that $\Lambda_\varphi^{\mathcal{N}}(\bar{x})$ is a Σ_1^0 subset of N . By Lemma 13, this Σ_1^0 -set is infinite. It follows from Lemma 5 that $\text{acl}(\bar{x})$, which is isomorphic to the k dimensional model of T , has a recursive presentation. \square

Our next goal is to extend this result also to the n -dimensional model of T . For what follows, it is convenient to note that not only is $\Lambda_\varphi^{\mathcal{N}}(X)$ recursively enumerable for a given X , but the sequence $(X_\varphi^i)_{i \in \omega}$ is uniformly recursive in X .

Lemma 15. *The set of triples (y, X, i) so that $y \in X_\varphi^i$, where X is a finite set given by canonical index, is recursive. It follows that the set of X for which $\Lambda_\varphi^{\mathcal{N}}(X)$ is finite is a recursively enumerable set.*

Proof. Since φ makes each coordinate algebraic over the others, that is $T \models \forall \bar{x} \exists^{< \infty} y \varphi(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_m)$, there is a uniform bound K so that $T \models \forall \bar{x} \exists^{< K} y \varphi(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_m)$. By model completeness, each formula $\exists^{=L} y \varphi(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_m)$ is a recursive set. Thus, from the finite set X_φ^i , we can determine for each tuple \bar{x} exactly how many y we need to find to include in X_φ^{i+1} . Thus it is recursive to find all such y , and thus to find X_φ^{i+1} . For the second statement, note that $\Lambda_\varphi^{\mathcal{N}}(X)$ is finite if and only if there exists an i so that $X_\varphi^i = X_\varphi^{i+1}$, which is a Σ_1^0 condition. \square

Lemma 16. *Assume \mathcal{N} is a recursive model of T of dimension $\geq n + 1$. Let $X \subseteq \mathcal{N}$ be an algebraically closed subset of dimension n . Then if X is infinite, it contains an infinite Σ_1^0 set.*

Proof. If there is some finite $B \subseteq X$ with $\Lambda_\varphi^{\mathcal{N}}(B)$ infinite, this is the needed infinite Σ_1^0 set, so we may assume there is no such B .

Fix \bar{b} to be a basis for X . By Lemma 13, $\Lambda_\varphi^{\mathcal{N}}(\bar{b}a)$ is infinite for any $a \notin X$. However, for any $a \in X$, $\Lambda_\varphi^{\mathcal{N}}(\bar{b}a)$ is finite. Thus Lemma 15 shows that X itself is Σ_1^0 as it is the set of a so that $\Lambda_\varphi^{\mathcal{N}}(\bar{b}a)$ is finite. \square

We can now extend our result to the dimension n model as well.

Theorem 17. *Suppose that \mathcal{N} is a finite dimensional recursive model of T . Then every model of T with dimension $k \in [n, \dim(\mathcal{N})]$ has a recursive presentation.*

Proof. For $k \geq n + 1$, this was shown in Theorem 14. For the case of dimension n , we again fix \bar{x} a generic n -tuple in \mathcal{N} and Lemma 6 shows that $\text{acl}(\bar{x})$ is Δ_2^0 . Since there is assumed to be a model of dimension n , $\text{acl}(\bar{x})$ is infinite and Lemma 16 shows that it contains an infinite Σ_1^0 set. Then Lemma 5 shows that $\text{acl}(\bar{x})$, which is isomorphic to the n -dimensional model of T , has a recursive presentation. \square

6. RELATING CIRCUIT SIZE TO MODEL DIMENSIONS

Thus far we have shown that for a model complete flat strongly minimal theory, if a finite dimensional model is recursive, then all the models of smaller dimension, down to n (the dimension of the smallest circuit) are also recursively presentable. We now consider how close this gets us to showing that all models of smaller dimension are recursively presentable. To do this, we let p be the dimension of the prime model of T and we will give some relationships between n and p .

Lemma 18. $p \leq n + 1$

Proof. By Lemma 13 and Observation 12, the closure of a generic $n + 1$ -tuple is infinite. Thus Fact 4 shows that there is an $n + 1$ -dimensional model of T . \square

Observation 19. *If $\mathcal{N} \models T$ is of dimension less than n , then $G_{\mathcal{N}}$ is disintegrated. In particular, if such a model \mathcal{N} exists, by the pigeonhole principle, the closure of every generic point in a model of T is infinite.*

Lemma 20. *If $n > 3$, then $p \geq n$.*

Proof. Assume to the contrary that $n > 3$, but $p < n$. Let \mathcal{M}_{n-1} be the model of T of dimension $n - 1$, which exists by Fact 4, and let b_1, \dots, b_{n-1} be a basis for \mathcal{M}_{n-1} . By Observation 19, the pregeometry $G_{\mathcal{M}_{n-1}}$ is disintegrated, and the closure of every point in \mathcal{M}_{n-1} is infinite. In particular, in $\text{acl}(b_{n-1})$ there is a generic enough point c such that

$$\mathcal{M}_{n-1} \models \exists x \varphi(b_1, \dots, b_{n-1}, c, x).$$

Choose such a c , and $d \in \varphi(b_1, \dots, b_{n-1}, c, \mathcal{M}_{n-1})$. Since $d \in \text{acl}(b_1, \dots, b_{n-1}, c) = \text{acl}(b_1, \dots, b_{n-1})$, by disintegration and independence of b_1, \dots, b_{n-1} , either $d \in \text{acl}(\emptyset)$ or there is a unique i such that $d \in \text{acl}(b_i)$. In particular, because $n > 3$, there is some $i' \neq n - 1$ such that $d \in \text{acl}(b_1, \dots, b_{i'-1}, b_{i'+1}, \dots, b_{n-1})$. Without loss of generality, $i' = 1$.

By choice of φ in Definition 11, we know $\varphi(\mathcal{M}_{n-1}, b_2, \dots, b_{n-1}, c, d)$ is finite, so b_1 is not generic over $b_2, \dots, b_{n-1}, c, d$. But $c, d \in \text{acl}(b_2, \dots, b_{n-1})$, so $b_1 \in \text{acl}(b_2, \dots, b_{n-1})$, in contradiction to b_1, \dots, b_{n-1} being a basis. \square

Remark 21. The proof of Lemma 20 can be modified slightly to show that if $n = 3$, then $p \geq 1$. Observe that if $p = 0$, then c can be chosen from $\text{cl}(\emptyset)$. In that case, the proof goes through also for $i' = n - 1$, because unlike before,

$c \in \text{acl}(b_1, \dots, b_{i'-1}, b_{i'+1}, \dots, b_{n-1})$ regardless of choice of i' . Thus, we only need $n \geq 3$ to get a contradiction, implying that if $n = 3$, then $p > 0$.

We now extend Lemma 20 to the case $n = 3$. Note that in this Lemma, we use flatness in an essential way. We do not know if the result holds without the assumption of flatness.

Lemma 22. *If $n = 3$, then $p \geq n$.*

Proof. We fix $n = 3$. Assume, towards a contradiction, that the model of dimension 2 exists. The type of an independent tuple in a strongly minimal theory is not dependent on the dimension of the model, so by Observation 19, the closure of every independent point is infinite, and $\text{acl}(ab) = \text{acl}(a) \cup \text{acl}(b)$ for any independent pair a, b .

Let $\psi(x_1, \dots, x_6) := \varphi(x_1, x_2, x_5, x_6) \wedge \varphi(x_3, x_4, x_5, x_6)$.

Claim 1. If a, b, c, d are independent in some $\mathcal{M} \models T$, then $\mathcal{M} \models \exists x_5, x_6 \psi(a, b, c, d, x_5, x_6)$.

Proof. By homogeneity, whenever x, y, z are independent there exists $w \in \varphi(\mathcal{M}, x, y, z)$ such that $\{w, x, y, z\}$ is a circuit. Take d, e, f independent, and let c be such that $\mathcal{M} \models \varphi(c, d, e, f)$ and $\{c, d, e, f\}$ is a circuit. Now take b independent from c, d, e, f and let a be such that $\mathcal{M} \models \varphi(a, b, e, f)$ and $\{a, b, e, f\}$ is a circuit.

We claim that $\{a, b, c, d\}$ are independent. Assume not, i.e., $a \in \text{acl}(b, c, d)$. By $n = 3$, either a is interalgebraic with a single element from $\{b, c, d\}$, or $\{a, b, c, d\}$ is a circuit. By construction, $a \notin \text{acl}(b)$. If $a \in \text{acl}(cd) \subseteq \text{acl}(cdef)$, then $b \in \text{acl}(aef) \subseteq \text{acl}(cdef)$, so this is also not the case. Thus $\{a, b, c, d\}$ is a circuit. Denote $E_1 = \text{acl}(abcd)$, $E_2 = \text{acl}(abef)$, $E_3 = \text{acl}(cdef)$. The sets E_1, E_2, E_3 are distinct – the union of any two is of dimension $d(abcdef) = 4$, whereas $d(E_i) = 3$ for each $i \in \{1, 2, 3\}$. Thus, $d(E_i \cap E_j) = 2$ for any i, j distinct. As for $d(E_1 \cap E_2 \cap E_3)$, note $E_1 \cap E_2 \cap E_3 = (E_1 \cap E_3) \cap (E_2 \cap E_3) = \text{acl}(cd) \cap \text{acl}(ef) = (\text{acl}(c) \cup \text{acl}(d)) \cap (\text{acl}(e) \cup \text{acl}(f)) = \text{acl}(\emptyset)$. So $d(E_1 \cap E_2 \cap E_3) = 0$.

Executing a flatness calculation for $\Sigma = \{E_1, E_2, E_3\}$ yields

$$d(abcdef) \leq \Delta(\Sigma) = 3 \cdot 3 - 3 \cdot 2 + 0 = 3$$

which contradicts independence of $bdef$. Thus, a, b, c, d are independent.

As $\mathcal{M} \models \exists x_5, x_6 \psi(a, b, c, d, x_5, x_6)$ and the type of an independent 4-tuple in a model of T is unique, this proves the claim. \square

Fix a, b, c independent in some $\mathcal{M} \models T$. By Claim 1, $\exists x_5, x_6 \psi(a, b, c, \mathcal{M}, x_5, x_6)$ is co-finite, so intersects $\text{acl}(a)$. Let $a' \in \text{acl}(a)$ be such that $\mathcal{M} \models \exists x_5, x_6 \psi(a, b, c, a', x_5, x_6)$. Now $\exists x_5, x_6 \psi(a, b, \mathcal{M}, a', x_5, x_6)$ is co-finite, because c is independent of a, b . In particular, we may take $a'' \in \text{acl}(a)$ such that $\mathcal{M} \models \exists x_5, x_6 \psi(a, b, a'', a', x_5, x_6)$.

Claim 2. There are $t_0, t_1 \in \text{acl}(ab)$ such that $\mathcal{M} \models \psi(a, b, a'', a', t_0, t_1)$.

Proof. If there are only finitely many t_0 satisfying $\exists x_6 \psi(a, b, a'', a', t_0, x_6)$, then any such t_0 is in $\text{acl}(ab)$. Otherwise, by strong minimality, there are co-finitely many such t_0 and we may choose $t_0 \in \text{acl}(ab)$ satisfying $\exists x_6 \psi(a, b, a'', a', t_0, x_6)$. With such a t_0 , $\psi(a, b, a'', a', t_0, \mathcal{M}) \subseteq \varphi(a, b, t_0, \mathcal{M}) \subseteq \text{acl}(ab)$. So, for any t_1 satisfying $\psi(a, b, a'', a', t_0, t_1)$, we have $t_0, t_1 \in \text{acl}(ab)$. \square

Take t_0, t_1 such as in Claim 2. Assume for a moment $t_i \in \text{acl}(a)$ for some $i \in \{0, 1\}$. Then $\varphi(a'', a', t_0, t_1)$ implies $t_{1-i} \in \text{acl}(a)$. Now $\varphi(a, b, t_0, t_1)$ implies $b \in \text{acl}(a)$, in contradiction. Therefore, it must be that $t_0, t_1 \notin \text{acl}(a)$, i.e. $t_0, t_1 \in \text{acl}(b)$. But now $a \in \text{acl}(bt_0t_1) = \text{acl}(b)$, a contradiction.

We have thus shown that there is no 2-dimensional model. It follows from Fact 4 that $p \geq 3 = n$. \square

7. RECURSION THEORETIC CONSEQUENCES

We now restate Theorem 17 in the case $n \neq 2$:

Theorem 23. *Let T be a model complete flat strongly minimal theory. Suppose further that the smallest circuit in a saturated model of T has size ≥ 4 (i.e. $n \geq 3$). Suppose that \mathcal{N} is a recursive model of finite dimension k . Then any model of T of dimension $\leq k$ has a recursive presentation.*

Proof. Theorem 17 tells us that any model of dimension $\leq k$ has a recursive presentation, given that its dimension is $\geq n$. But Lemmas 20 and 22 show that there are no models of dimension $< n$. \square

We now focus on the remaining case of $n = 2$. It follows from Theorem 17 that if any model of finite dimension ≥ 2 is recursive, then the 2-dimensional model is recursive. Yet it is possible that the 0- and 1-dimensional models exist. Our methods which exploit the pregeometry to find elements in the closure of a tuple are ill equipped for this case. In particular, for the target models, the pregeometry is disintegrated!

In this case, we are not able to use the assumption of a recursive 2-dimensional model to prove that the 0- and 1-dimensional models have recursive presentations, but we are able to do so assuming the existence of a recursive 3-dimensional model.

Theorem 24. *Suppose that T is a model complete flat strongly minimal theory. Further suppose that the smallest circuit in a saturated model of T has length 3 (i.e. $n = 2$). Suppose that there is a recursive model of T of finite dimension ≥ 3 . Then the 0- and 1-dimensional models of T have recursive presentations, if they exist.*

Proof. By Theorem 14, let \mathcal{N} be a recursive model of T of dimension 3. Fix a, b, c a basis for \mathcal{N} .

Claim 1. If $\text{acl}(a)$ is infinite, it contains an infinite Σ_1^0 set.

Proof. For any finite $X \subseteq \text{acl}(a)$, the set $\Lambda_\varphi^\mathcal{N}(X)$ is Σ_1^0 , so we may assume it is finite.

Case 1: There is some $a_0 \in \text{acl}(a)$ such that $\Lambda_\varphi^\mathcal{N}(baa_0)$ is infinite.

The element b represents the generic behavior over aa_0 . Then for every d , the set $\Lambda_\varphi^\mathcal{N}(daa_0)$ is finite if and only if $d \in \text{acl}(aa_0) = \text{acl}(a)$. By Lemma 15, $\text{acl}(a)$ is a Σ_1^0 set.

Case 2: For every $a_0 \in \text{acl}(a)$, the set $\Lambda_\varphi^\mathcal{N}(baa_0)$ is finite.

By Lemma 13, $\Lambda_\varphi^\mathcal{N}(bac)$ is infinite. Thus, for every d , if the set $\Lambda_\varphi^\mathcal{N}(bad)$ is finite, then $d \in \text{acl}(ba)$. Similarly, if $\Lambda_\varphi^\mathcal{N}(cad)$ is finite, then $d \in \text{acl}(ca)$.

We find that $d \in \text{acl}(a)$ implies $\Lambda_\varphi^\mathcal{N}(bad)$ is finite, and by genericity of b, c over a , also $\Lambda_\varphi^\mathcal{N}(cad)$ is finite. On the other hand, if both $\Lambda_\varphi^\mathcal{N}(bad)$ and $\Lambda_\varphi^\mathcal{N}(cad)$ are finite, then $d \in \text{acl}(ba) \cap \text{acl}(ca)$. It follows that $d \in \text{acl}(a)$, as otherwise by exchange

$b, c \in \text{acl}(ad)$ contradicting $d(abc) > 2$. Again, Lemma 15 shows that $\text{acl}(a)$ is a Σ_1^0 set. \square

Claim 2. If $\text{acl}(\emptyset)$ is infinite, it contains an infinite Σ_1^0 set.

Proof. For any finite $X \subseteq \text{acl}(\emptyset)$, the set $\Lambda_\varphi^{\mathcal{N}}(X)$ is Σ_1^0 , so we may assume it is finite.

Case 1: There are some $e, e' \in \text{acl}(\emptyset)$ such that $\Lambda_\varphi^{\mathcal{N}}(aee')$ is infinite.

The element a represents the generic behavior over ee' . Then for every d , the set $\Lambda_\varphi^{\mathcal{N}}(dee')$ is finite if and only if $\{d, e, e'\} \subseteq \text{acl}(\emptyset)$. Thus Lemma 15 shows that $\text{acl}(\emptyset)$ is Σ_1^0 .

Case 2: Case 1 fails and there is $e \in \text{acl}(\emptyset)$ such that $\Lambda_\varphi^{\mathcal{N}}(aeb)$ is infinite.

The element b represents the generic behavior over ae . So for each point e' , if the set $\Lambda_\varphi^{\mathcal{N}}(aee')$ is finite then $e' \in \text{acl}(ae) = \text{acl}(a)$. Similarly, if $\Lambda_\varphi^{\mathcal{N}}(bee')$ is finite then $e' \in \text{acl}(be) = \text{acl}(b)$. We now argue that $\text{acl}(\emptyset)$ is exactly the set of e' so that both $\Lambda_\varphi^{\mathcal{N}}(aee')$ and $\Lambda_\varphi^{\mathcal{N}}(bee')$ are finite. Since we are not in case 1, every $e' \in \text{acl}(\emptyset)$ is so that both $\Lambda_\varphi^{\mathcal{N}}(aee')$ and $\Lambda_\varphi^{\mathcal{N}}(bee')$ are finite. On the other hand, if both $\Lambda_\varphi^{\mathcal{N}}(aee')$ and $\Lambda_\varphi^{\mathcal{N}}(bee')$ are finite, then $e' \in \text{acl}(a) \cap \text{acl}(b)$, but $\text{acl}(a) \cap \text{acl}(b) = \text{acl}(\emptyset)$ since a and b are mutually generic. Thus Lemma 15 again shows that $\text{acl}(\emptyset)$ is Σ_1^0 .

Case 3: Both Case 1 and Case 2 fail.

By Lemma 13, we know that if $\Lambda_\varphi^{\mathcal{N}}(aeb)$ is finite, then $e \in \text{acl}(ab)$. So if the three sets $\Lambda_\varphi^{\mathcal{N}}(aeb)$, $\Lambda_\varphi^{\mathcal{N}}(aec)$, $\Lambda_\varphi^{\mathcal{N}}(bec)$ are finite, then $e \in \text{acl}(ab) \cap \text{acl}(ac) \cap \text{acl}(bc)$. But by exchange and genericity of the tuple abc , $\text{acl}(ab) \cap \text{acl}(bc) = \text{acl}(b)$. Thus $\text{acl}(ab) \cap \text{acl}(ac) \cap \text{acl}(bc) = \text{acl}(a) \cap \text{acl}(b) \cap \text{acl}(c) = \text{acl}(\emptyset)$ (again by exchange and genericity). On the other hand, because Case 2 fails (and Case 1 fails), whenever $e \in \text{acl}(\emptyset)$, then the three sets are finite. Thus Lemma 15 again shows that $\text{acl}(\emptyset)$ is Σ_1^0 . \square

If a 0- or 1-dimensional model $\mathcal{N}' \prec \mathcal{N}$ exists, then the appropriate claim guarantees it contains a Σ_1^0 set. Thus, Lemma 5 shows that \mathcal{N}' has a recursive presentation. \square

We now prove our main theorem.

Theorem 1. If T is a flat model complete strongly minimal theory, then $\text{SRM}(T)$ is contained in one of the following schema:

- $[0, \alpha)$ for $\alpha \in \omega + 2$, $[0, n] \cup \{\omega\}$ for $n \in \omega$, $\{\omega\}$,
- $\{1\}$, $\{2\}$, $\{0, 2\}$, $\{1, 2\}$, $\{1, \omega\}$, $\{2, \omega\}$, $\{0, 2, \omega\}$, $\{1, 2, \omega\}$.

Further, the sets in the first line are in fact spectra of flat model complete strongly minimal theories. We do not know whether any of the 8 sets (not schema) in the second line are spectra of flat model complete strongly minimal theories.

Proof. In the case where $n \neq 2$, Theorem 23 shows that $\text{SRM}(T) \cap [0, \omega)$ is an initial segment of $[0, \omega)$. Thus $\text{SRM}(T)$ is either of the form $[0, \alpha)$ for some $\alpha \in \omega + 2$ or $[0, n] \cup \{\omega\}$ for some $n \in \omega$. Next, we consider the case where $n = 2$. By Theorem 17, the collection of dimensions of recursive models of T is initial among finite dimensions ≥ 2 . Let us first assume that $\text{SRM}(T) \cap [3, \omega) \neq \emptyset$. Then in particular, there is a recursive model of finite dimension ≥ 3 . Then Theorem 24 shows that the 0 and 1-dimensional models (if they exist) are recursive as well. Thus the collection of dimensions of recursive models of T is initial among finite dimensions. Thus

again $\text{SRM}(T)$ is either of the form $[0, \alpha)$ for some $\alpha \in \omega + 2$ or $[0, n] \cup \{\omega\}$ for some $n \in \omega$.

Now we consider the case where $n = 2$ and $[3, \omega) \cap \text{SRM}(T) = \emptyset$. There are 16 such subsets of $[0, \omega]$. Of these, \emptyset , $\{0\}$, $\{0, 1\}$, $\{0, \omega\}$, $\{0, 1, 2\}$, $\{0, 1, \omega\}$, $\{0, 1, 2, \omega\}$, and $\{\omega\}$ are already covered by schemata in the first item. The remaining 8 are the sets in the second item.

The fact that the sets in the schema $[0, \alpha)$ for $\alpha \in \omega + 2$ are spectra of strongly minimal theories with finite signatures is proved in [And11a]. In that paper, it is not established that the theory is model complete and has a flat pregeometry, but the construction fits into the framework in [AM19, Section 2] and [AM19, Corollary 2.23 and Lemma 3.20] show that the theory is model complete and has a flat pregeometry.

Similarly, the fact that the set $\{\omega\}$ is a spectrum of a strongly minimal theory with a finite signature is proved in [And11b]. Once again, this construction fits into the framework in [AM19, Section 2]¹ and so the theory of $\mathcal{M}|_{L'}$ (see [And11b, Lemma 27]) is flat and model complete. From there, the model \mathcal{M}' is formed by restricting $\mathcal{M}|_{L'}$ to the signature $\{R\}$. Each other symbol from L' is existentially definable in this restriction by [And11b, Lemma 23]. They are also universally definable because, for every x, y there must be exactly two z so that $R_i(x, y, z)$ holds (since μ allows this many and $xy \leq \mathcal{M}$). Thus the reduct to \mathcal{M}' is also flat and model complete. The theory of \mathcal{M}' is shown to have spectrum $\{\omega\}$ in [And11b, Theorem 30].

Finally, the sets in the schema $[0, n] \cup \{\omega\}$ for $n \in \omega$ are shown to be spectra of flat model complete strongly minimal theories in [AM19]. \square

REFERENCES

- [AM14] Uri Andrews and Alice Medvedev. Recursive spectra of strongly minimal theories satisfying the Zilber trichotomy. *Trans. Amer. Math. Soc.*, 366(5):2393–2417, 2014.
- [AM19] Uri Andrews and Omer Mermelstein. $[0, n] \cup \{\omega\}$ is a spectrum of a non-disintegrated flat strongly minimal model complete theory in a language with finite signature. *Submitted, preprint at arXiv:1908.09387*, 2019.
- [And11a] Uri Andrews. New spectra of strongly minimal theories in finite languages. *Ann. Pure Appl. Logic*, 162(5):367–372, 2011.
- [And11b] Uri Andrews. A new spectrum of recursive models using an amalgamation construction. *J. Symbolic Logic*, 76(3):883–896, 2011.
- [And13] Uri Andrews. The degrees of categorical theories with recursive models. *Proc. Amer. Math. Soc.*, 141(7):2501–2514, 2013.
- [And14] U. Andrews. Spectra of recursive models of disintegrated strongly minimal theories. *Lobachevskii J. Math.*, 35(4):287–291, 2014.
- [BL71] J. T. Baldwin and A. H. Lachlan. On strongly minimal sets. *J. Symbolic Logic*, 36:79–96, 1971.
- [GHL⁺03] Sergey S. Goncharov, Valentina S. Harizanov, Michael C. Laskowski, Steffen Lempp, and Charles F. D. McCoy. Trivial, strongly minimal theories are model complete after naming constants. *Proc. Amer. Math. Soc.*, 131(12):3901–3912, 2003.
- [Gon78] S. S. Gončarov. Constructive models of \aleph_1 -categorical theories. *Mat. Zametki*, 23(6):885–888, 1978.
- [HKS06] Denis R. Hirschfeldt, Bakhadyr Khoussainov, and Pavel Semukhin. An uncountably categorical theory whose only computably presentable model is saturated. *Notre Dame J. Formal Logic*, 47(1):63–71, 2006.

¹There is a slight difference between the papers regarding the definition of an extension being “of the form” of another extension, but this makes no difference. In fact, the construction in [And11b] can be altered to use the definition from [AM19] with no change to the rest of the proof.

- [Hru93] Ehud Hrushovski. A new strongly minimal set. *Ann. Pure Appl. Logic*, 62(2):147–166, 1993. Stability in model theory, III (Trento, 1991).
- [KNS97] Bakhadyr Khoushainov, Andre Nies, and Richard A. Shore. Computable models of theories with few models. *Notre Dame J. Formal Logic*, 38(2):165–178, 1997.
- [Nie99] André Nies. A new spectrum of recursive models. *Notre Dame J. Formal Logic*, 40(3):307–314, 1999.

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